# Difference equations for correlation functions of Belavin's $\mathbb{Z}_n$-symmetric model with boundary reflection

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**Abstract**

Belavin's $\mathbb{Z}_n$-symmetric elliptic model with boundary reflection is considered on the basis of the boundary CTM bootstrap. We find non-diagonal $K$-matrices for $n > 2$ that satisfy the reflection equation (boundary Yang–Baxter equation), and also find non-diagonal Boltzmann weights for the $A^{(1)}_{n-1}$-face model even for $n \geqq 2$. We derive difference equations of the quantum Knizhnik-Zamolodchikov type for correlation functions of the boundary model. The boundary spontaneous polarization is obtained by solving the simplest difference equations. The resulting quantity is the square of the spontaneous polarization for the bulk $\mathbb{Z}_n$-symmetric model, up to a phase factor.

## 1 Introduction

Integrable models with a boundary have been studied in massive quantum theories [1, 2, 3, 4, 5, 6] and half infinite lattice models [7, 8, 9, 10, 11, 12, 13]. The boundary interaction is specified by the boundary $S$-matrix for massive quantum theories [2], and by the reflection matrix $K$ for lattice models [7]. The integrability in the presence of reflecting boundary is ensured by the reflection equation (boundary Yang–Baxter equation) [1], in addition to the Yang–Baxter equation for bulk (i.e., without boundary) theory [14].

It was shown in [2] that the boundary vacuum of boundary integrable theories can be expressed in terms of the vacuum and the creation operators in the bulk theory. In [8] the explicit bosonic formulae of the boundary vacuum of the boundary XXZ model were obtained by using the bosonization of the vertex operators associated with the bulk XXZ model [15].

The quantum Knizhnik-Zamolodchikov equations [16, 17] are satisfied by both correlation functions and form factors for bulk field theories [18] and for bulk lattice models [19, 20] with the affine quantum group symmetry. It is shown in [9] that correlation functions and form factors in semi-infinite XXZ/XYZ spin chains with integrable boundary conditions satisfy the boundary analogue of the quantum Knizhnik-Zamolodchikov equation [1].

In this paper we study Belavin's $\mathbb{Z}_n$-symmetric vertex model [21] with integrable boundary condition, the boundary Belavin model. The $R$-matrix of Belavin's model is expressed in terms of elliptic functions of the spectral parameter $z$ so that the $R$-matrix has doubly quasi periodicity. Thus we expect that the $K$-matrix of the boundary Belavin model also possesses appropriate transformation properties with respect to $z$ compatible to those of the $R$-matrix. We shall show that under such assumption the $K$-matrix of the boundary Belavin model is inevitably non-diagonal for $n > 2$. Our solution is diagonal for $n = 2$ but different from the one used in [9].

On the basis of boundary CTM bootstrap [14, 19, 9] we find that the correlation functions for the boundary Belavin model satisfy a set of difference equations, the boundary analogue of the quantum Knizhnik–Zamolodchikov equation. Furthermore, by solving the simplest difference equations, we obtain the boundary spontaneous polarization which turns out to be the square of that for the bulk $\mathbb{Z}_n$-symmetric model [22].

The rest of this paper is organized as follows. In section 2 we review Belavin's $\mathbb{Z}_n$-symmetric model, thereby fixing our notations. In section 3 we give two non-diagonal solutions to the reflection equation, one is a constant $K$-matrix, and the other is an elliptic $K$-matrix. Furthermore, we consider the boundary analogue of the vertex-face correspondence to discuss the connection between our $K$-matrix and the boundary weights of the $A^{(1)}_{n-1}$ model [23]. In section 4 we construct lattice realization of the boundary vacuum states and vertex operators from the boundary CTM bootstrap approach. In section 5 we derive difference equations for $N$-point functions of the boundary Belavin model. We solve the simplest difference equations with $N = 1$ for free boundary condition to obtain the explicit expression of the boundary spontaneous polarization. The result gives the higher rank generalization of that for the boundary eight vertex model [9]. In section 6 we summarize the results obtained in this paper, and give some concluding remarks.

## 2 Belavin's vertex model and the reflection equation

### 2.1 Elliptic theta functions

For a complex number $\tau$ in the upper half-plane, let $\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau$ be the lattice generated by 1 and $\tau$, and $E_\tau := \mathbb{C}/\Lambda_\tau$ the complex torus which can be identified with an elliptic curve. For $a, b \in \mathbb{R}$, introduce the Jacobi theta function

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) := \sum_{m\in\mathbb{Z}} \exp\left\{\pi\sqrt{-1}(m + a)\ \ [(m + a)\tau + 2(z + b)]\right\}. \tag{2.1}$$

Hereafter a positive integer $n \geq 2$ is fixed and we will use the following compact symbols

$$\sigma^{(n)}_{\boldsymbol{\alpha}}(z) = \vartheta \begin{bmatrix} \alpha_2/n + 1/2 \\ \alpha_1/n + 1/2 \end{bmatrix} (z, \tau), \qquad \theta^{(j)}_n(z) = \vartheta \begin{bmatrix} 1/2 - j/n \\ 1/2 \end{bmatrix} (z, n\tau), \tag{2.2}$$

for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{Z} \otimes \mathbb{Z}$ and for $j \in \mathbb{Z}_n$; and

$$h(z) := \prod_{j=0}^{n-1} \theta^{(j)}(z) / \prod_{j=1}^{n-1} \theta^{(j)}(0).$$

The superscript $(n)$ and the subscript $n$ will be often suppressed when we have no fear of confusion.

The elliptic theta functions are expressed in terms of the product series

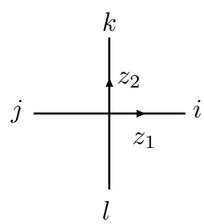
$$\begin{aligned} \theta^{(j)}(z) &= \sqrt{-1}\omega^{j/2} t^{n(1/2-j/n)^2} u^{-1+2j/n} (t^{2n}; t^{2n})_\infty (t^{2j} u^2; t^{2n})_\infty (t^{2(n-j)} u^{-2}; t^{2n})_\infty, \\ h(z) &= t^{(n-1)/4} \frac{(t^{2n}; t^{2n})^3_\infty}{(t^2; t^2)^3_\infty} \sigma_{\mathbf{0}}(z, \tau) = \sqrt{-1} t^{n/4} \frac{(t^{2n}; t^{2n})^3_\infty}{(t^2; t^2)^2_\infty} u^{-1} (u^2; t^2)_\infty (t^2 u^{-2}; t^2)_\infty, \end{aligned} \tag{2.3}$$

where

$$(a; q_1, \cdots, q_k)_\infty := \prod_{m_1=0}^{\infty} \cdots \prod_{m_k=0}^{\infty} (1 - a q_1^{m_1} \cdots q_k^{m_k}).$$

## 2.2 Belavin's vertex model

Let $V = \mathbb{C}^n$ and $\{v_i\}_{i \in \mathbb{Z}_n}$ be the standard orthonormal basis of $V$ with the inner product $(v_j, v_k) = \delta_{jk}$. Let $V_z$ be a copy of $V$ with a spectral parameter $z$. The $\mathbb{Z}_n$-Baxter model is a vertex model on a two-dimensional square lattice $\mathcal{L}$ such that the state variables take on values of $\mathbb{Z}_n$-spin. Each oriented line of $\mathcal{L}$ carries a spectral parameter varying from line to line. We assign a $\mathbb{Z}_n$-valued local state on each edge. Let

$R(z_1 - z_2)^{ik}_{jl} :=$ 


be a local Boltzmann weight for a single vertex with bond states $i, j, k, l \in \mathbb{Z}_n$. Arrows denotes orientations of lines. We now define the linear map on $V_{z_1} \otimes V_{z_2}$ called the $R$-matrix as follows:

$$R^{V_{z_1}, V_{z_2}}(v_j \otimes v_l) = \sum_{i,k \in \mathbb{Z}_n} (v_i \otimes v_k) R(z_1 - z_2)^{ik}_{jl}.$$

Belavin [21] considered the $\mathbb{Z}_n$-symmetric model satisfying

$$\begin{aligned} &\text{(i)} \quad R(z)^{ik}_{jl} = 0, \quad \text{unless } i + k = j + l, \ \text{ mod } n, \\ &\text{(ii)} \quad R(z)^{i+pk+p}_{j+pl+p} = R(z)^{ik}_{jl}, \quad \text{for every } i, j, k, l \text{ and } p \in \mathbb{Z}_n. \end{aligned} \tag{2.4}$$

In terms of two linear map in $V$

$$g v_i = \omega^i v_i, \qquad h v_i = v_{i-1}, \tag{2.5}$$

where $\omega = \exp(2\pi\sqrt{-1}/n)$, the conditions (2.4) can be rephrased as follows:

$$\begin{aligned} R(z)(g \otimes g) &= (g \otimes g) R(z), \\ R(z)(h \otimes h) &= (h \otimes h) R(z). \end{aligned} \tag{2.6}$$

Thus the $R$-matrix of Belavin's $\mathbb{Z}_n$-symmetric model is of the form

$$R(z) = \frac{1}{\kappa(z)}\overline{R}(z), \qquad \overline{R}(z) = \sum_{\boldsymbol{\alpha}\in G_n} u_{\boldsymbol{\alpha}}(z) I_{\boldsymbol{\alpha}} \otimes I_{\boldsymbol{\alpha}}^{-1}. \tag{2.7}$$

Here $G_n = \mathbb{Z}_n \otimes \mathbb{Z}_n$, and $I_{\boldsymbol{\alpha}} = g^{\alpha_1} h^{\alpha_2}$ for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. The normalization factor $\kappa(z)$ will be given lator. The coefficient function $u_{\boldsymbol{\alpha}}(z)$ is determined by imposing the $R$-matrix satisfies the Yang-Baxter equation

$$R_{12}(z_1 - z_2)R_{13}(z_1 - z_3)R_{23}(z_2 - z_3) = R_{23}(z_2 - z_3)R_{13}(z_1 - z_3)R_{12}(z_1 - z_2), \tag{2.8}$$

where $R_{ij}(z)$ denotes the matrix on $V^{\otimes 3}$, which acts as $R(z)$ on the $i$-th and $j$-th components and as identity on the other one. Belavin's solution to (2.8) is given as follows:

$$u_{\boldsymbol{\alpha}}(z) = u_{\boldsymbol{\alpha}}^{(n)}(z, w) := \frac{1}{n} \frac{\sigma_{\boldsymbol{\alpha}}(z + w/n)}{\sigma_{\boldsymbol{\alpha}}(w/n)}, \tag{2.9}$$

where $w(\neq 0 \bmod \Lambda_\tau)$ is a constant. It is obvious that the following initial condition holds:

$$\overline{R}(0) = P, \qquad P(x \otimes y) = y \otimes x. \tag{2.10}$$

In order to facilitate the derivation of the similar results for the $K$-matrix of the boundary $\mathbb{Z}_n$-symmetric model, we give brief sketches of proofs of several well known properties for Belavin's $R$-matrix.

**Proposition 2.1** *The Boltzmann weights or the elements of $R$-matrix are given as follows [24]:*

$$\overline{R}(z)_{jl}^{ik} = \begin{cases} \dfrac{h(z)\theta^{(i-k)}(z+w)}{\theta^{(j-k)}(z)\theta^{(i-j)}(w)} & \textit{if } i+k = j+l, \textit{ mod } n, \\ 0 & \textit{otherwise.} \end{cases} \tag{2.11}$$

[Proof] Because of the $\mathbb{Z}_n$-symmetry,

$$R_{0l-j}^{i-jk-j}(z) = R_{jl}^{ik}(z) = \sum_{\boldsymbol{\alpha}\in G_n} u_{\boldsymbol{\alpha}}(z)(I_{\boldsymbol{\alpha}})_j^i (I_{\boldsymbol{\alpha}}^{-1})_l^k = \delta_{j+l}^{i+k} \sum_{\alpha_1 \in \mathbb{Z}_n} u_{(\alpha_1, j-i)}(z)\omega^{(i-l)\alpha_1}.$$

Set $\mathcal{R}^{ab}(z) = R_{0a+b}^{ab}(z)$. Then we have

$$\mathcal{R}_n^{ab}(z, v) = \sum_{\alpha_1 \in \mathbb{Z}_n} u_{(\alpha_1, -a)}^{(n)}(z, v)\omega^{-b\alpha_1}. \tag{2.12}$$

The transformation property of $\mathcal{R}^{ab}(z)$ and the initial condition $\mathcal{R}^{ab}(0) = \delta^{b0}$ imply that

$$\mathcal{R}^{ab}(z) = 0, \qquad \text{at } z = c\tau \ (c \neq -b, \bmod n) \text{ and } z = (a-b)\tau - w, \bmod \Lambda_{n\tau}. \tag{2.13}$$

Hence $\mathcal{R}^{ab}(z)$ has the form

$$\mathcal{R}^{ab}(z) = C^{ab}(w)\theta^{(a-b)}(z+w) \prod_{c \neq -b} \theta^{(c)}(z).$$

By substituting $z = -b\tau$ we have

$$C^{ab}(w)^{-1} = \theta^{(a)}(w) \prod_{c \neq 0} \theta^{(c)}(0),$$

which concludes that (2.11) holds. $\Box$

As a corollary of Proposition 2.1 we have [24]

$$PR(-w) = -R(-w), \qquad R(w)P = R(w). \tag{2.14}$$

Now we assume that $0 < t < q < u < 1$, where $t := \exp(\pi\sqrt{-1}\tau), q := \exp(\pi\sqrt{-1}w)$, and $u := \exp(-\pi\sqrt{-1}z)$. Following Baxter [14] we call such domain of parameters the principal regime. Note that (2.11) is weights of the eight-vertex model when $n = 2$.

## 2.3 Unitarity and crossing symmetry

Belavin's $R$-matrix satisfies the unitarity and crossing symmetry relations [24, 25, 26].

**Proposition 2.2** *Belavin's $R$-matrix satisfies the following unitarity relation or the first inversion relation:*

$$\overline{R}_{21}(z)\overline{R}_{12}(-z) = \rho_1(z, w) I \otimes I, \tag{2.15}$$

*where*

$$\rho_1(z, w) = \frac{\sigma(z + w)\sigma(-z + w)}{\sigma^2(w)}. \tag{2.16}$$

[Proof] Note that

$$\begin{aligned}
\overline{R}_{21}(z)\overline{R}_{12}(-z) &= \sum_{\boldsymbol{\alpha} \in G_n} u^{(n)}_{\boldsymbol{\alpha}}(z, w) I_{\boldsymbol{\alpha}}^{-1} \otimes I_{\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta} \in G_n} u^{(n)}_{\boldsymbol{\beta}}(-z, w) I_{\boldsymbol{\beta}} \otimes I_{\boldsymbol{\beta}}^{-1} \\
&= \sum_{\boldsymbol{\alpha}\boldsymbol{\beta} \in G_n} u^{(n)}_{\boldsymbol{\alpha}}(z) u^{(n)}_{\boldsymbol{\beta}}(-z) I_{\boldsymbol{\alpha}}^{-1} I_{\boldsymbol{\beta}} \otimes I_{\boldsymbol{\alpha}} I_{\boldsymbol{\beta}}^{-1} \\
&= \sum_{\boldsymbol{a} \in G_n} f^{(n)}_{\boldsymbol{a}}(z, w) I_{\boldsymbol{a}} \otimes I_{\boldsymbol{a}}^{-1},
\end{aligned}$$

where

$$f^{(n)}_{\boldsymbol{a}}(z, w) = \sum_{\boldsymbol{\alpha} \in G_n} \omega^{\langle \boldsymbol{\alpha}, \boldsymbol{a} \rangle} u^{(n)}_{\boldsymbol{\alpha}}(z, w) u^{(n)}_{\boldsymbol{a}+\boldsymbol{\alpha}}(-z, w), \tag{2.17}$$

and $\langle \boldsymbol{\alpha}, \boldsymbol{a} \rangle = \alpha_1 a_2 - \alpha_2 a_1$. Proposition 2.2 is thus reduced to

$$f^{(n)}_{\boldsymbol{a}}(z, w) = \rho_1(z, w) \delta_{\boldsymbol{a}\boldsymbol{0}}. \tag{2.18}$$

Concerning the proof of (2.18), see Theorem 3.3 and Lemma 3.2 in [26]. $\Box$

Next we describe the crossing symmetry for Belavin's $\mathbb{Z}_n$-symmetric model. For that purpose let us recall the $R$-matrix on $K \otimes L$, where $K = V_{z_1} \otimes \cdots \otimes V_{z_k}$ and $L = V_{z'_1} \otimes \cdots \otimes V_{z'_l}$:

$$\begin{aligned} R^{K,V_{z'}} &:= R^{V_{z_1},V_{z'}}_{1;k+1} \cdots R^{V_{z_k},V_{z'}}_{k;k+1}, \\ R^{K,L} &:= R^{K,V_{z'_l}}_{1\cdots k;k+l} \cdots R^{K,V_{z'_1}}_{1\cdots k;k+1}. \end{aligned}$$

YBE holds for $R^{K,L}$ by virtue of YBE for $R^{V,V}$ (2.8)

$$R^{K,L}_{12} R^{K,M}_{13} R^{L,M}_{23} = R^{L,M}_{23} R^{K,M}_{13} R^{K,L}_{12}, \tag{2.19}$$

as a linear map on $K \otimes L \otimes M$.

For special $K^k_z = V_{z_1} \otimes \cdots \otimes V_{z_k}$ such that $z_j = z + (k+1-j)w$ $(1 \leqq j \leqq k)$, the fusion operator $\pi$ associated with $K^k_z$ is given as follows [27]:

$$\pi := R^{V_{z_1},V_{z_2}}_{k-1;k} R^{V_{z_1}\otimes V_{z_2},V_{z_3}}_{k-2,k-1;k} \cdots R^{V_{z_1}\otimes\cdots\otimes V_{z_{k-1}},V_{z_k}}_{1,\cdots,k-1;k}. \tag{2.20}$$

From the first equation of (2.14) and the Yang–Baxter equation (2.8) we have

$$\pi(K^k_z) = \Lambda^k(V) = \mathrm{Anti}(K^k_z). \tag{2.21}$$

Let $V^*$ be the dual space of $V$ and $\{v^*_i\}_{i\in\mathbb{Z}_n}$ be the dual basis of $\{v_i\}_{i\in\mathbb{Z}_n}$. Then we have the isomorphism $C : V^*_{z+nw/2} \longrightarrow \mathrm{Anti}(K^{n-1}_z)$

$$C v^*_i = \sum_{i_1,\cdots,i_{n-1}} \frac{\epsilon^{i_1\cdots i_{n-1}}_i}{\sqrt{(n-1)!}} v_{i_1} \otimes \cdots \otimes v_{i_{n-1}}, \tag{2.22}$$

where $\epsilon^{i_1\cdots i_{n-1}}_i$ is the $n$-th order completely antisymmetric tensor. The spectral parameter $z+nw/2$ associated with the dual space $V^*$ refers to the mean value of $n-1$ spectral parameters $z+(n-1)w, \cdots,$ $z+w$ of $V$[1]. Then the $R$-matrices on $V \otimes V^*$ and $V^* \otimes V$ are defined as follows:

$$\begin{aligned} R^{V_{z_1},V^*_{z_2+nw/2}} &= (I \otimes C)^{-1} R^{V_{z_1},V_{z_2+(n-1)w}\otimes\cdots\otimes V_{z_2+w}} (I \otimes C), \\ R^{V^*_{z_1+nw/2},V_{z_2}} &= (C \otimes I)^{-1} R^{V_{z_1+(n-1)w}\otimes\cdots\otimes V_{z_1+w},V_{z_2}} (C \otimes I). \end{aligned} \tag{2.23}$$

The un-normalized $\overline{R}$ on $V \otimes V^*$ and $V^* \otimes V$ are also defined in a similar manner.

**Proposition 2.3** *The R-matrix on $V \otimes V^*$ and $V^* \otimes V$ defined in (2.23) meet the crossing symmetry [25, 26]:*

$$\begin{aligned} \overline{R}^{V_{z_2},V^*_{z_1+nw/2}}_{21} &= (\overline{R}^{V_{z_1},V_{z_2}}_{12})^{t_1} \prod_{p=2}^{n-1} \frac{h(-z_1+z_2+pw)}{h(w)}, \\ \overline{R}^{V^*_{z_1+nw/2},V_{z_2}}_{12} &= (\overline{R}^{V_{z_2},V_{z_1+nw}}_{21})^{t_1} \prod_{p=1}^{n-2} \frac{h(-z_1+z_2-pw)}{h(w)}, \end{aligned} \tag{2.24}$$

*where $t_i$ denotes the transposition of the $i$-th space.*

[1] Note that the spectral parameter of $V^*$ is shifted by $nw/2$ from the one in [22, 26].

[Proof] Let

$$\overline{R}_{21}^{V_{z_2},V^*_{z_1+nw/2}}(v_j\otimes v_l^*)=\sum_{i,k}(v_i\otimes v_k^*)\,a_{jl}^{ik}(-z_1+z_2)$$

Because of the initial condition (2.10) and the second equation of (2.14), the element $a_{jl}^{ik}(-z)$ vanishes at $-z=pw$, where $p=2,\cdots,n-1$. Thus we have an entire function $b_{jl}^{ik}(-z)$ from $a_{jl}^{ik}(-z)$ devided by $h(-z-2w)\cdots f(-z-(n-1)w)$.

The transformation property of $b_{jl}^{ik}(-z)$ are the same as $\overline{R}_{kj}^{li}(z)$. It follows from the second equation of (2.14) that $b_{jl}^{ik}(-z)=0$ at $z=c\tau$ for $c\neq j-k$ and at $z=(i-k)\tau-w$, which coincide the zeros of $\overline{R}_{kj}^{li}(z)$ (2.13). Thus $b_{jl}^{ik}(-z)$ equals $\check{R}_{ij}^{kl}(z)$ up to a scalar factor, which is determined by substituting $z=(k-i)\tau$. The second equation of (2.24) can be shown in a similar way. □

From (2.16) and (2.24), we have the following second inversion relation [24, 26]

$$\sum_{jl}\overline{R}_{12}^{t_1}(z)\overline{R}_{21}^{t_1}(-z-nw)=\rho_2(z,w)I, \tag{2.25}$$

where

$$\rho_2(z,w)=\frac{h(-z)h(z+nw)}{h^2(w)}. \tag{2.26}$$

Imposing the unitarity and crossing symmetry condition with respect to the normalized $R$-matrix:

$$R_{21}(z)R_{12}(-z)=I\otimes I, \tag{2.27}$$

$$R_{21}^{V_{z_2},V^*_{z_1+nw/2}}=(R_{12}^{V_{z_1},V_{z_2}})^{t_1},\qquad R_{12}^{V^*_{z_1+nw/2},V_{z_2}}=(R_{21}^{V_{z_2},V_{z_1+nw}})^{t_1}, \tag{2.28}$$

the normalization factor $\kappa(z)$ should obey the following functional equations:

$$\begin{aligned}\kappa(z)\kappa(-z)&=\rho_1(z,w),\\ \kappa(z)\kappa(-z-nw)&=\rho_2(z,w).\end{aligned} \tag{2.29}$$

Hereafter $\kappa(z)$ is often denoted by $\kappa(u)$ through the relation $u=\exp(-\pi\sqrt{-1}z)$. In the principal regime using (2.3) the following expression solves (2.29) [24]

$$\kappa(u)=u^{-(n-2)/n}\frac{(u^2;t^2)_\infty(t^2u^{-2};t^2)_\infty}{(q^2;t^2)_\infty(t^2q^{-2};t^2)_\infty}\bar{\kappa}(u), \tag{2.30}$$

where

$$\bar{\kappa}(u)=\frac{(q^2u^2;t^2,q^{2n})_\infty(q^{2n}u^{-2};t^2,q^{2n})_\infty(t^2q^{-2}u^2;t^2,q^{2n})_\infty(t^2q^{2n}u^{-2};t^2,q^{2n})_\infty}{(q^{2+2n}u^{-2};t^2,q^{2n})_\infty(u^2;t^2,q^{2n})_\infty(t^2q^{-2+2n}u^{-2};t^2,q^{2n})_\infty(t^2u^2;t^2,q^{2n})_\infty}.$$

From $\kappa(1)=1$ the initial condition for $R$ also holds:

$$R(0)=P. \tag{2.31}$$

# 3 Boundary Belavin model

## 3.1 Reflection equation for the boundary Belavin model

In this section we consider the following reflection equation or the boundary Yang–Baxter equation:

$$K_2(z_2)R_{21}(z_1+z_2)K_1(z_1)R_{12}(z_1-z_2) = R_{21}(z_1-z_2)K_1(z_1)R_{12}(z_1+z_2)K_2(z_2). \tag{3.1}$$

The reflection equation (3.1) is valid when $z_1 = z_2$ because $R(0) = P$. Furthermore, the following Lemma holds:

**Lemma 3.1** *The reflection equation* (3.1) *is valid when* (1) $z_1 = 0$; (2) $z_1 = -z_2$ *provided*

$$\begin{aligned} &(1)\ \textit{Boundary initial condition}: \quad K(0) = I;\\ &(2)\ \textit{Boundary unitarity relation}: \quad K(z)K(-z) = I, \end{aligned} \tag{3.2}$$

*respectively.*

[Proof] It is evident from the unitarity (2.27) and the initial condition (2.31) for $R$-matrix. □

Here we notice that Belavin's $R$-matrix have the following quasi-periodic properties

$$\begin{aligned} \overline{R}(z+1) &= -(g\otimes I)^{-1}\overline{R}(z)(g\otimes I) = (I\otimes g)\overline{R}(z)(I\otimes g)^{-1},\\ \overline{R}(z+\tau) &= -(h\otimes I)^{-1}\overline{R}(z)(h\otimes I)\times \exp\left\{-2\pi\sqrt{-1}\left(z+\tfrac{\tau}{2}+\tfrac{w}{n}\right)\right\}\\ &= -(I\otimes h)\overline{R}(z)(I\otimes h)^{-1}\times \exp\left\{-2\pi\sqrt{-1}\left(z+\tfrac{\tau}{2}+\tfrac{w}{n}\right)\right\}. \end{aligned} \tag{3.3}$$

Thus we have the following Proposition:

**Proposition 3.2** *Let*

$$K(z) = \frac{1}{\lambda(z)}\overline{K}(z),$$

*where $\lambda(z)$ is a scalor function. Suppose (3.2) and the following quasi transformation property:*

$$\begin{aligned} \overline{K}(z+1) &= -g\overline{K}(z)g,\\ \overline{K}(z+\tau) &= -h\overline{K}(z)h\times \exp\left\{-2\pi\sqrt{-1}\left(z+\tfrac{\tau}{2}+c\right)\right\}, \end{aligned} \tag{3.4}$$

*where $c$ is a constant. Then $\overline{K}(z)$ solves (3.1).*

[Proof] Let $F(z_1,z_2)$ stand for the difference of the LHS and the RHS of (3.1). Then we have

$$\begin{aligned} F(z_1+1,z_2) &= -(g\otimes I)F(z_1,z_2)(g\otimes I),\\ F(z_1+\tau,z_2) &= -(h\otimes I)F(z_1,z_2)(h\otimes I)\times\exp(-2\pi\sqrt{-1}B), \end{aligned} \tag{3.5}$$

where $B = 3z_1 + 3\tau/2 + 2w/n + c$. The second equation of (3.5) implies that the $(ik,jl)$-th element of $F(z_1,z_2)$ satisfies

$$F(z_1+\tau,z_2)^{ik}_{jl} = -F(z_1+\tau,z_2)^{i+1k}_{j-1l}\times(-2\pi\sqrt{-1}B). \tag{3.6}$$

Thus we find that $F(p\tau,z_2)^{ik}_{jl} \propto F(0,z_2)^{i+pk}_{j-pl} = 0$ for $0 \leqq p \leqq n-1$ from Lemma 3.1. Similarly, we have $F(z_2+p\tau,z_2)^{ik}_{jl} = F(-z_2+p\tau,z_2)^{ik}_{jl} = 0$ for $0 \leqq p \leqq n-1$:

$$F(p\tau,z_2)^{ik}_{jl} = F(z_2+p\tau,z_2)^{ik}_{jl} = F(-z_2+p\tau,z_2)^{ik}_{jl} = 0, \quad (0 \leqq p \leqq n-1). \tag{3.7}$$

Assume that $F(z_1+\tau,z_2)^{ik}_{jl}$ is not identically zero. From Richey-Tracy's lemma (see section 3 in [24] or Lemma 2.4 in [26]) we conclude that $F(z_1+\tau,z_2)^{ik}_{jl}$ has $3n$ zeros in $E_{n\tau}$ whose sum is equal to $nc-2w-3n(n-1)\tau-(i+j)\tau$. The contradiction to (3.7) implies the claim of this Proposition. □

## 3.2 Solutions of the reflection equation

Under the assumption of the quasi periodicity (3.4) compatible to (3.3) we find that the $K(z)$ is not a diagonal matrix for $n>2$. When $n=2$ we can take $K(z)$ diagonal because of $g^{-1}=g$ and $h^{-1}=h$. The most general and non-diagonal solution for $n=2$ is given in [28, 29]. Other non-diagonal solutions for $D^{(2)}_n$-vertex model are given in [30].

In this paper we consider the following two solutions of (3.1), which can be also found in [31].

### 3.2.1 Constant $K$-matrix

**Proposition 3.3** *Let*

$$\mathcal{K}_0 v_j = v_{n-j}, \tag{3.8}$$

*where $v_n = v_0$. Then $\mathcal{K}_0$ solves (3.1).*

[Proof] It is easy to see $g\mathcal{K}_0 g = h\mathcal{K}_0 h = \mathcal{K}_0$. Hence we have

$$\begin{aligned}
& K_2(z_2)R_{21}(z_1+z_2)K_1(z_1)R_{12}(z_1-z_2) \\
= \; & I\otimes\mathcal{K}_0 \sum_{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}}(z_1+z_2)(I_{\boldsymbol{\alpha}}^{-1}\otimes I_{\boldsymbol{\alpha}})(\mathcal{K}_0\otimes I)\sum_{\boldsymbol{\beta}} u_{\boldsymbol{\beta}}(z_1-z_2)(I_{\boldsymbol{\beta}}\otimes I_{\boldsymbol{\beta}}^{-1}) \\
= \; & \mathcal{K}_0\otimes\mathcal{K}_0 \sum_{\boldsymbol{\alpha}} \omega^{\alpha_1\alpha_2} u_{\boldsymbol{\alpha}}(z_1+z_2) I_{\boldsymbol{\alpha}}\otimes I_{\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}} \omega^{\beta_1\beta_2} u_{\boldsymbol{\beta}}(z_1-z_2) I_{\boldsymbol{\beta}}\otimes I_{-\boldsymbol{\beta}} \\
= \; & \mathcal{K}_0\otimes\mathcal{K}_0 \sum_{\boldsymbol{\beta}} \omega^{\beta_1\beta_2} u_{\boldsymbol{\beta}}(z_1-z_2) I_{\boldsymbol{\beta}}\otimes I_{-\boldsymbol{\beta}} \sum_{\boldsymbol{\alpha}} \omega^{\alpha_1\alpha_2} u_{\boldsymbol{\alpha}}(z_1+z_2) I_{\boldsymbol{\alpha}}\otimes I_{\boldsymbol{\alpha}} \\
= \; & \sum_{\boldsymbol{\beta}} \omega^{\beta_1\beta_2} u_{\boldsymbol{\beta}}(z_1-z_2)(I_{-\boldsymbol{\beta}}\otimes I_{\boldsymbol{\beta}})(\mathcal{K}_0\otimes I) \sum_{\boldsymbol{\alpha}} \omega^{\alpha_1\alpha_2} u_{\boldsymbol{\alpha}}(z_1+z_2)(I_{\boldsymbol{\alpha}}\otimes I_{-\boldsymbol{\alpha}})(I\otimes\mathcal{K}_0) \\
= \; & R_{21}(z_1-z_2)K_1(z_1)R_{12}(z_1+z_2)K_2(z_2),
\end{aligned}$$

that implies this Proposition. □

### 3.2.2 Elliptic $K$-matrix

Let

$$m = \begin{cases} n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even,} \end{cases}$$

and let

$$\mathcal{K}(z) = \sum_{\boldsymbol{\alpha}\in G_m} \omega^{2\alpha_1\alpha_2} u^{(n)}_{2\boldsymbol{\alpha}}(z,v) I_{2\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}\in G_m} u^{(n)}_{2\boldsymbol{\alpha}}(z,v) J_{\boldsymbol{\alpha}}, \tag{3.9}$$

where

$$J_{\boldsymbol{\alpha}} = h^{\alpha_2} g^{2\alpha_1} h^{\alpha_2}$$

for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, and $v(\neq 0 \bmod \Lambda_\tau)$ is a constant. Using the identity

$$\frac{1}{m}\sum_{\alpha_1=0}^{m-1} \omega^{2\alpha_1(i-\alpha_2)} = \begin{cases} \delta_{\alpha_2,i} & \text{if } n \text{ is odd,} \\ \delta_{\alpha_2,i} + \delta_{\alpha_2,i-m} & \text{if } n \text{ is even,} \end{cases}$$

we have $\mathcal{K}(0) = \mathcal{K}_0$.

**Lemma 3.4** *The following quasi transformation property holds:*

$$\begin{aligned} \mathcal{K}(z+1) &= -g^{-1}\mathcal{K}(z)g; \\ \mathcal{K}(z+\tau) &= -h^{-1}\mathcal{K}(z)h \times \exp\left\{-2\pi\sqrt{-1}\left(z + \tfrac{\tau}{2} + \tfrac{v}{m}\right)\right\}. \end{aligned} \tag{3.10}$$

[Proof] This is based on the transformation properties of the elliptic theta function. $\square$

**Lemma 3.5** *Let $\overline{K}(z) = \mathcal{K}_0\mathcal{K}(z)$. Then the boundary inversion relation holds:*

$$\overline{K}(z)\overline{K}(-z) = \rho_1(z,v)I. \tag{3.11}$$

[Proof] Direct calculation shows

$$\begin{aligned} \overline{K}(z)\overline{K}(-z) &= \mathcal{K}_0 \sum_{\boldsymbol{\alpha}\in G_m} u^{(n)}_{2\boldsymbol{\alpha}}(z,v) J_{\boldsymbol{\alpha}} \mathcal{K}_0 \sum_{\boldsymbol{\beta}\in G_m} u^{(n)}_{2\boldsymbol{\beta}}(-z,v) J_{\boldsymbol{\beta}} \\ &= \sum_{\boldsymbol{\alpha}\in G_m} u^{(n)}_{2\boldsymbol{\alpha}}(z,v) J_{-\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}\in G_m} u^{(n)}_{2\boldsymbol{\beta}}(-z,v) J_{\boldsymbol{\beta}} \\ &= \sum_{\boldsymbol{\alpha}\in G_m}\sum_{\boldsymbol{\beta}\in G_m} \omega^{2\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle} u^{(n)}_{2\boldsymbol{\alpha}}(z,v) u^{(n)}_{2\boldsymbol{\beta}}(-z,v) J_{\boldsymbol{\alpha}-\boldsymbol{\beta}} \\ &= \sum_{\boldsymbol{a}\in G_m} g^{(n)}_{\boldsymbol{a}}(z,v) I_{\boldsymbol{a}}, \end{aligned}$$

where

$$g^{(n)}_{\boldsymbol{a}}(z,v) = \sum_{\boldsymbol{\alpha}\in G_m} \omega^{2\langle\boldsymbol{\alpha},\boldsymbol{a}\rangle} u^{(n)}_{2\boldsymbol{\alpha}}(z,v) u^{(n)}_{2(\boldsymbol{a}+\boldsymbol{\alpha})}(-z,v). \tag{3.12}$$

By comparing $g^{(n)}_{\boldsymbol{a}}(z,v)$ with $f^{(n)}_{\boldsymbol{a}}(z,w)$ defined in (2.17), we easily have $g^{(n)}_{\boldsymbol{a}}(z,v) = f^{(m)}_{\boldsymbol{a}}(z,v)$ and hence (3.11) holds for even $n$. Repeating the similar argument in Proposition 2.2 we can also obtain (3.11) for odd $n$. $\square$

**Theorem 3.6** *Let $\overline{K}(z) = \mathcal{K}_0\mathcal{K}(z)$. Then $\overline{K}(z)$ solves the reflection equation (3.1).*

[Proof] From Lemma 3.4 we find that $\overline{K}(z)$ satisfies (3.4) with $c = v/m$. Since $\overline{K}(0) = \mathcal{K}_0{}^2 = I$, the $\overline{K}(z)$ also satisfies the first equation of (3.2). It follows from Lemma 3.5 that $\overline{K}(z)$ satisfies the second one of (3.2). Thus $\overline{K}(z)$ is a solution to the reflection equation (3.1) from Proposition 3.2. □

**Remark.** Our $K$-matrix for $n = 2$ is different from the one used in [9] so that the readers should be careful to compare our results with those of $n = 2$.

### 3.3 Matrix elements of $K$-matrix

In this subsection we calculate the $(j, k)$-th element of $\overline{K}(z)$:

$$\overline{K}(z)v_k = \sum_{j\in\mathbb{Z}_n} v_j \overline{K}(z)^j_k.$$

Note that

$$\overline{K}(z)^j_k = \mathcal{K}(z)^{n-j}_k = \sum_{\alpha_2\in\mathbb{Z}_m} \delta^{2\alpha_2}_{j+k} \sum_{\alpha_1\in\mathbb{Z}_m} u^{(n)}_{(2\alpha_1,j+k)}(z, v)\omega^{-(j-k)\alpha_1}$$

When $n$ is even, thanks to the sum over $\alpha_2$, $\overline{K}(z)^j_k = 0$ if $j + k$ is odd. By comparing (2.12) we obtain

$$\overline{K}(z)^j_k = \begin{cases} \mathcal{R}_m^{-\frac{j+k}{2},\frac{j-k}{2}}(z, v) & \text{if } j + k \text{ is even}, \\ 0 & \text{if } j + k \text{ is odd}, \end{cases} \tag{3.13}$$

for even $n$,

$$\overline{K}(z)^j_k = \begin{cases} \mathcal{R}_n^{-j-k,\frac{j-k}{2}}(z, v) & j - k \text{ is even}, \\ \mathcal{R}_n^{-j-k,\frac{j-k+n}{2}}(z, v) & j - k \text{ is odd}, \end{cases} \tag{3.14}$$

for odd $n$.

We are now in a position to determine the normalization factor $\lambda(z)$. The boundary inversion relation (3.11) implies

$$\lambda(z)\lambda(-z) = \rho_1(z, v). \tag{3.15}$$

Furthermore, the boundary crossing symmetry holds for $n = 2$ [7, 2, 8, 9]:

$$K(z)^j_k = \sum_{j',k'} R(-2z - w)^{j'\,1-k'}_{1-j\,k} K(-z - w)^{k'}_{j'}, \tag{3.16}$$

which implies that

$$\frac{\lambda(-z - w)}{\lambda(z)} = \frac{1}{\bar{\kappa}(u^2)} \frac{(q^2u^{-2}; t^2)_\infty(t^2q^{-2}u^2; t^2)_\infty}{(u^2; t^2)_\infty(t^2u^{-2}; t^2)_\infty}. \tag{3.17}$$

Since $V^* \cong \Lambda^{n-1}(V) \not\cong V$ for $n > 2$, the LHS of (3.16) for higher $n$ should be replaced by the $(j, k)$-th element of the dual $K$-matrix. We wish to discuss this point again in section 4.

Here we assume the following functional relation holds for $n \geqq 2$:

$$\frac{\lambda(-z - \frac{n}{2}w)}{\lambda(z)} = \frac{1}{\bar{\kappa}(u^2)} \frac{(q^nu^{-2}; t^2)_\infty(t^2q^{-n}u^2; t^2)_\infty}{(u^2; t^2)_\infty(t^2u^{-2}; t^2)_\infty}. \tag{3.18}$$

Not (3.18) but (3.15) is important to calculate the spontaneous polarization in section 5, so that we proceed further under the assumption (3.18). By solving (3.15) and (3.17) we obtain

$$\lambda(z) = \frac{1}{(r^2;t^2)_\infty (t^2r^{-2};t^2)_\infty} \frac{(r^2u^2;t^2,q^{2n})_\infty (t^2r^{-2}u^2;t^2,q^{2n})_\infty}{(r^2q^{2n}u^{-2};t^2,q^{2n})_\infty (t^2r^{-2}q^{2n}u^{-2};t^2,q^{2n})_\infty} \frac{\phi(u^2)}{\phi(u^{-2})}, \tag{3.19}$$

where $r = \exp(-\pi\sqrt{-1}v)$, and

$$\begin{aligned} \phi(x) &= \frac{(q^nx;t^2,q^{2n})_\infty (t^2q^nx;t^2,q^{2n})_\infty}{(q^{2n}x;t^2,q^{2n})_\infty (t^2x;t^2,q^{2n})_\infty (r^2q^nx;t^2,q^{2n})_\infty (t^2r^{-2}q^nx;t^2,q^{2n})_\infty} \\ &\times \frac{(q^{2n+2}x^2;t^2,q^{4n})_\infty (t^2q^{2n-2}x^2;t^2,q^{4n})_\infty}{(q^{2n}x^2;t^2,q^{4n})_\infty (t^2q^{2n}x^2;t^2,q^{4n})_\infty}. \end{aligned}$$

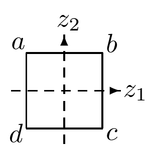
### 3.4 Comments on boundary weights for the boundary $A^{(1)}_{n-1}$ face model

In this subsection we wish to discuss the boundary analogue of the vertex-face correspondence. Concerning the case $n = 2$, see [13, 33]. Let us consider the bulk $A^{(1)}_{n-1}$-face model whose local state takes on values of $P$, the weight lattice of $A^{(1)}_{n-1}$ [32]. An ordered pair $(a, b) \in P^2$ is called admissible if $b = a + \hat{j}$, for a certain $j \in \mathbb{Z}_n$, where

$$\hat{j} = v_j - \frac{1}{n}\sum_{k=0}^{n-1} v_k.$$

Let

$$W\left(\begin{array}{cc|c} a & b & \\ d & c & z_1 - z_2 \end{array}\right) =$$




be the local Boltzmann weight for a state configuration $(a, b, c, d)$ round a face. Then $W\left(\begin{array}{cc|c} a & b & \\ d & c & z \end{array}\right) = 0$ unless all the four pairs $(a, b), (a, d), (b, c)$ and $(d, c)$ are admissible. Non-zero Boltzmann weights are given as follows:

$$W\left(\begin{array}{cc|c} a & b & \\ d & c & z \end{array}\right) = \frac{1}{w(z,w)} \overline{W}\left(\begin{array}{cc|c} a & b & \\ d & c & z \end{array}\right), \tag{3.20}$$

where $w(z, w)$ is a scalar function and

$$\begin{aligned} \overline{W}\left(\begin{array}{cc|c} a & a+\hat{j} & \\ a+\hat{j} & a+2\hat{j} & z \end{array}\right) &= \frac{h(z+w)}{h(w)}, \\ \overline{W}\left(\begin{array}{cc|c} a & a+\hat{j} & \\ a+\hat{j} & a+\hat{j}+\hat{k} & z \end{array}\right) &= \frac{h(a_{jk}w - z)}{h(a_{jk}w)} \quad (j \neq k), \\ \overline{W}\left(\begin{array}{cc|c} a & a+\hat{k} & \\ a+\hat{j} & a+\hat{j}+\hat{k} & z \end{array}\right) &= \frac{h(z)}{h(w)} \frac{h(a_{jk}w + w)}{h(a_{jk}w)} \quad (j \neq k). \end{aligned} \tag{3.21}$$

Here

$$a_{jk} = \bar{a_j} - \bar{a_k}, \qquad \bar{a_j} = (a+\rho, v_j),$$

and $\rho = \sum_{j=0}^{n-1}(n-1-j)\hat{j}$ is the half sum of the positive roots.

Jimbo, Miwa and Okado [32] introduced the intertwining vectors to show the equivalence between the $\mathbb{Z}_n$-symmetric model and the $A^{(1)}_{n-1}$ model. Let

$$\begin{aligned} t^a_b(z) :=& \quad {}^t(t^{a(0)}_b(z), \cdots, t^{a(n-1)}_b(z)), \\ t^{a(i)}_b(z) :=& \begin{cases} \theta^{(i)}(z+\delta - nw\bar{a}_j) & \text{if } b = a+\bar{\epsilon}_j, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{3.22}$$

where $\delta$ is an arbitrary constant. Then we have the so-called vertex-face correspondence [32]:

$$\overline{R}(z_1-z_2)t^a_d(z_1)\otimes t^d_c(z_2) = \sum_b \overline{W}\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| z_1-z_2\right) t^b_c(z_1)\otimes t^a_b(z_2). \tag{3.23}$$

Thanks to (3.23) the Boltzmann weights (3.21) solve the face-type Yang-Baxter equation [32]:

$$\begin{aligned} &\sum_g \overline{W}\left(\begin{array}{cc} b & c \\ g & d \end{array}\middle| z_1-z_2\right)\overline{W}\left(\begin{array}{cc} a & b \\ f & g \end{array}\middle| z_1-z_3\right)\overline{W}\left(\begin{array}{cc} f & g \\ e & d \end{array}\middle| z_2-z_3\right) \\ =& \sum_g \overline{W}\left(\begin{array}{cc} a & b \\ g & c \end{array}\middle| z_2-z_3\right)\overline{W}\left(\begin{array}{cc} g & c \\ e & d \end{array}\middle| z_1-z_3\right)\overline{W}\left(\begin{array}{cc} a & g \\ f & e \end{array}\middle| z_1-z_2\right). \end{aligned} \tag{3.24}$$

Let us now consider the boundary $A^{(1)}_{n-1}$-face model. By analogy with the bulk case, we find the following Proposition:

**Proposition 3.7** *Assume that the existence of boundary weights $V$'s satisfying*

$$\overline{K}(z)t^a_c(z) = \sum_b \overline{V}\left(a \begin{array}{c} b \\ c \end{array}\middle| z\right) t^a_b(-z). \tag{3.25}$$

*Then $\overline{V}$ solves the face-type reflection equation*

$$\begin{aligned} &\sum_{b,e} \overline{V}\left(f \begin{array}{c} g \\ e \end{array}\middle| z_2\right)\overline{W}\left(\begin{array}{cc} a & f \\ b & e \end{array}\middle| z_1+z_2\right)\overline{V}\left(b \begin{array}{c} e \\ c \end{array}\middle| z_1\right)\overline{W}\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| z_1-z_2\right) \\ =& \sum_{b,e} \overline{W}\left(\begin{array}{cc} a & f \\ b & g \end{array}\middle| z_1-z_2\right)\overline{V}\left(b \begin{array}{c} g \\ e \end{array}\middle| z_1\right)\overline{W}\left(\begin{array}{cc} a & b \\ d & e \end{array}\middle| z_1+z_2\right)\overline{V}\left(d \begin{array}{c} e \\ c \end{array}\middle| z_2\right). \end{aligned} \tag{3.26}$$

In order to solve (3.25), let us recall the dual intertwining vectors [34, 26, 25]

$$\begin{aligned} t^{*b}_a(z) :=& \quad (t^{*b}_{a(0)}(z), \cdots, t^{*b}_{a(n-1)}(z)), \\ t^{*a+\hat{j}}_{a(i)}(z) :=& \quad (\tilde{\Phi}^a(z))^i_j / \det \Phi^a(z). \end{aligned} \tag{3.27}$$

Here $\Phi^a(z)$ is a matrix whose $(i,j)$-component is $t^{a(i)}_{a+\hat{j}}(z)$, and $\tilde{\Phi}^a(z)$ is a cofactor matrix of $\Phi^a(z)$. Note that $t^a_b(z)$ is a column vector while $t^{*b}_a(z)$ is a row vector. Thus by the rule of multiplication of matrices, $t^{*b}_a(z)t^c_d(z')$ represents a scalar function while $t^a_b(z)t^{*c}_d(z')$ does a function-valued matrix. Since $t^a_b(z)$ and $t^{*b}_a(z)$ enjoy the following orthogonal properties

$$t^{*a+\hat{j}}_a(z)t^a_{a+\hat{k}}(z) = \delta_{jk}, \tag{3.28}$$

$$\sum_{j=0}^{n-1} t^a_{a+\bar{\epsilon}_j}(z)t^{*a+\hat{j}}_a(z) = I_n, \tag{3.29}$$

the boundary analogue of the vertex-face correspondence (3.25) is equivalent to

$$\overline{V}\left(a \begin{array}{c} b \\ c \end{array} \middle| z\right) = t^{*b}_a(-z)K(z)t^a_c(z) = \sum_{j,k} t^{*b}_{a(j)}(-z)K(z)^j_k t^{a(k)}_c(z). \tag{3.30}$$

**Proposition 3.8** *Let*

$$V\left(a \begin{array}{c} b \\ c \end{array} \middle| z\right) = \frac{1}{\lambda(z)}\overline{V}\left(a \begin{array}{c} b \\ c \end{array} \middle| z\right),$$

*where $\lambda(z)$ is the same scalar function as for $K(z)$, and $\overline{V}$ is defined by (3.30). Then the boundary weights $V$'s satisfy the initial condition*

$$V\left(a \begin{array}{c} b \\ c \end{array} \middle| 0\right) = \delta^b_c, \tag{3.31}$$

*and the inversion relation*

$$\sum_g V\left(a \begin{array}{c} b \\ g \end{array} \middle| z\right) V\left(a \begin{array}{c} g \\ c \end{array} \middle| -z\right) = \delta^b_c. \tag{3.32}$$

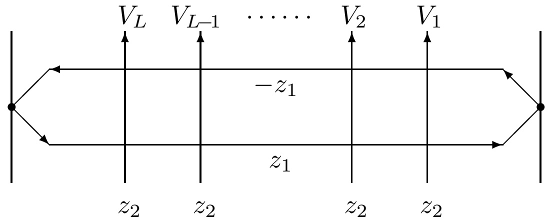
[Proof] The initial condition (3.31) follows from that for $K(z)$ and (3.28). The inversion relation (3.32) follows from (3.29), (3.11) and (3.28). □

The boundary weights $V\left(a \begin{array}{c} b \\ c \end{array} \middle| z\right)$ are non-diagonal in the sense that they do not vanish even for $b \neq c$ as a function of $z$. Hence (3.30) does not coincide with the diagonal solution of (3.26) involving the bulk Boltzmann weights for the $A^{(1)}_{n-1}$-face model given in [23] for $n \geqq 2$. Such disagreement indicates that there may exist unknown solution to (3.1) corresponding to the solution given in [23] and also unknown solution to (3.26) corresponding to our $K$-matrix, throughout the boundary vertex-face correspondence.

### 3.5 Commuting transfer matrix

The transfer matrix with $L$ columns,

$T_L(z_1, z_2) :=$




is expressed in terms of $R$ and $K$-matrices as follows [7]:

$$\begin{aligned} T_L(z_1, z_2) &= \mathrm{Tr}_0\, K_+(z_1)\mathcal{T}(z_1, z_2) \\ \mathcal{T}(z_1, z_2) &= \mathrm{Tr}_0\, \mathcal{T}(-z_1 - z_2)^{-1} K_-(z_1)\mathcal{T}(z_1 - z_2). \end{aligned} \tag{3.33}$$

Here

$$\begin{aligned} \mathcal{T}(z_1 - z_2) &= R_{01}^{V_{z_1}, V_{z_2}} \cdots R_{0L}^{V_{z_1}, V_{z_2}} \in \mathrm{End}\,(V_0 \otimes V_1 \otimes \cdots \otimes V_L), \\ \mathcal{T}(-z_1 - z_2)^{-1} &= R_{L0}^{V_{z_2}, V_{-z_1}} \cdots R_{10}^{V_{z_2}, V_{-z_1}} \in \mathrm{End}\,(V_0 \otimes V_1 \otimes \cdots \otimes V_L), \end{aligned}$$

are monodromy matrices satisfying

$$R_{12}(z_1 - z_2)\mathcal{T}_1(z_1)\mathcal{T}_2(z_2) = \mathcal{T}_2(z_2)\mathcal{T}_1(z_1)R_{12}(z_1 - z_1'), \tag{3.34}$$

and $\mathrm{T}r_0$ signifies the trance on the auxiliary space associated with the spectral parameters $z_1$ and $-z_1$. Note that the boundary monodromy matrix $\mathcal{T}(z, z')$ is a solution to the reflection equation:

$$\mathcal{T}_2(z_1', z_2)R_{21}(z_1 + z_2)\mathcal{T}_1(z_1, z_2)R_{12}(z_1 - z_2) = R_{21}(z_1' - z_1)\mathcal{T}_1(z_1, z_2)R_{12}(z_1 + z_1')\mathcal{T}_2(z_1', z_2). \tag{3.35}$$

**Proposition 3.9** *If one takes*

$$K_-(z) = K(z, v), \quad K_+(z) = K(-z - \tfrac{n}{2}w, v') \in \mathit{End}\,(V_0), \tag{3.36}$$

*where $v$ and $v'$ are arbitrary parameters, the transfer matrices (3.33) commute each other [7]:*

$$[T_L(z_1, z_2), T_L(z_1', z_2)] = 0. \tag{3.37}$$

[Proof] From the crossing symmetry (2.28) and the unitarity (2.27) we have

$$\begin{aligned} & T_L(z_1, z_2)T_L(z_1', z_2) \\ = \; & \mathrm{Tr}_1 K_1(-z_1 - \tfrac{n}{2}w)\mathcal{T}_1(z_1, z_2)\, \mathrm{Tr}_2 K_2(-z_1' - \tfrac{n}{2}w)\mathcal{T}_2(z_1', z_2) \\ = \; & \mathrm{Tr}_1\mathrm{Tr}_2 K_2(-z_1' - \tfrac{n}{2}w)K_1^{t_1}(-z_1 - \tfrac{n}{2}w)\mathcal{T}_1^{t_1}(z_1, z_2)\mathcal{T}_2(z_1', z_2) \\ = \; & \mathrm{Tr}_1\mathrm{Tr}_2 K_2(-z_1' - \tfrac{n}{2}w)K_1^{t_1}(-z_1 - \tfrac{n}{2}w)R_{21}^{t_1}(-z_1 - z_2 - nw)R_{12}^{t_1}(z_1 + z_2)\mathcal{T}_1^{t_1}(z_1, z_2)\mathcal{T}_2(z_1', z_2) \\ = \; & \mathrm{Tr}_1\mathrm{Tr}_2 K_2(-z_1' - \tfrac{n}{2}w)(R_{21}(-z_1 - z_2 - nw)K_1(-z_1 - \tfrac{n}{2}w))^{t_1}(\mathcal{T}_1(z_1, z_2)R_{12}(z_1 + z_2))^{t_1}\mathcal{T}_2(z_1', z_2) \\ = \; & \mathrm{Tr}_1\mathrm{Tr}_2 K_2(-z_1' - \tfrac{n}{2}w)R_{21}(-z_1 - z_2 - nw)K_1(-z_1 - \tfrac{n}{2}w)R_{12}(z_2 - z_1) \\ \times \; & R_{21}(z_1 - z_2)\mathcal{T}_1(z_1, z_2)R_{12}(z_1 + z_2)\mathcal{T}_2(z_1', z_2), \end{aligned}$$

where we use $\mathrm{Tr}\, AB = \mathrm{Tr}\, A^t B^t$. Furthermore, from (3.35) we have

$$
\begin{aligned}
&= \mathrm{Tr}_1\mathrm{Tr}_2 R_{21}(z_2 - z_1)K_1(-z_1 - \tfrac{n}{2}w)R_{12}(-z_1 - z_2 - nw)K_2(-z_1' - \tfrac{n}{2}w) \\
&\times\ \mathcal{T}_2(z_1', z_2)R_{21}(z_1 + z_2)\mathcal{T}_1(z_1, z_2)R_{12}(z_1 - z_2) \\
&= \mathrm{Tr}_1\mathrm{Tr}_2 K_1(-z_1 - \tfrac{n}{2}w)(K_2(-z_1' - \tfrac{n}{2}w)R_{12}(-z_1 - z_2 - nw))^{t_2}(R_{21}(z_1 + z_2)\mathcal{T}_2(z_1', z_2))^{t_2}\mathcal{T}_1(z_1, z_2) \\
&= \mathrm{Tr}_1\mathrm{Tr}_2 K_1(-z_1 - \tfrac{n}{2}w)(R_{12}(-z_1 - z_2 - nw)K_2(-z_1' - \tfrac{n}{2}w))^{t_2}(\mathcal{T}_2(z_1', z_2)R_{21}(z_1 + z_2))^{t_2}\mathcal{T}_1(z_1, z_2) \\
&= \mathrm{Tr}_1\mathrm{Tr}_2 K_1(-z_1 - \tfrac{n}{2}w)K_2^{t_2}(-z_1' - \tfrac{n}{2}w)\mathcal{T}_2^{t_2}(z_1', z_2)\mathcal{T}_1(z_1, z_2) \\
&= T_L(z_1', z_2)T_L(z_1, z_2),
\end{aligned}
$$

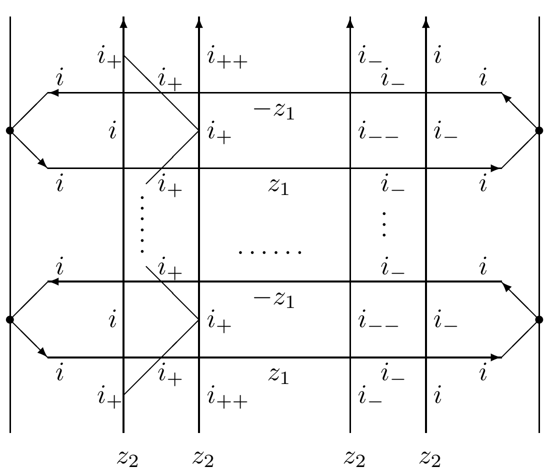
that implies the commutativity (3.37). □

# 4 Boundary CTM bootstrap

In this section we construct lattice realization of vertex operators and the boundary vacuum states for the boundary Belavin model.

## 4.1 Partition function

Let us consider the inhomogeneous lattice $\mathcal{L}_{LM}$ with $2M$ horizontal lines carrying alternating spectral parameters $z_1$ and $-z_1$ and $L(\equiv 0 \bmod n)$ vertical lines carrying the spectral parameter $z_2$ as below:




The lattice $\mathcal{L}_{LM}$ and the $i$-th ground state. The arrows stand for the orientation of the spectral parameters. The dots •'s stand for the boundary interaction $K(z)$. For the sake of simplicity, we here denote the state $i \pm 1$ and $i \pm 2$ by $i_\pm$ and $i_{\pm\pm}$, respectively. A zigzag line on which the state variables take $i + 1$ is presented for transparency.

In this paper we restrict ourselves to the principal regime $0 < t < q < r < u_\pm < 1$, where $u_\pm = \exp\left(-\pi\sqrt{-1}(z_1 \pm z_2)\right)$. In this regime of parameters, the bulk Boltzmann weights of the type $R(z)^{j+1,j}_{j,j+1}$ dominates the others; and the boundary Boltzmann weight $K^i_i(z)$ is the largest among $K^j_i(z)$ for fixed $i$. Thus in the low temperature limit $t, q \to 0$, only the configuration such that the spin variables take the same value along the zigzag line (see the above figure) and increase by one in the direction from West to East, is possible. We call it a configuration of the ground state labeled by the boundary state $i \in \mathbb{Z}_n$. Actually, the boundary weight $K^0_0(z)$ (and $K^m_m(z)$ if $n$ is even) are the largest among $K^i_i(z)$. We therefore have only one real ground state for odd $n$ and two for even $n$. Nevertheless, we regard all $n$ kinds of configurations as the ground states.

In what follows, we fix one of them (say, $i$) and define all the correlation functions in terms of the low-temperature series expansion (i.e., the formal power series of $t$ and $q$). Then the lowest order of them comes from the $i$-th ground state configuration. Furthermore, any finite order contribution is derived from the configurations which differ from that of the $i$-th ground state by altering a finite number of spins. It is equivalent to taking the GNS representation obtained from the $i$-th ground state ($i$-th GNS representation) as the Hilbert space. It is expected that the correlation function defined in such a way is an analytic function which has a finite convergence radius if there exists the phase transition at a finite temperature.

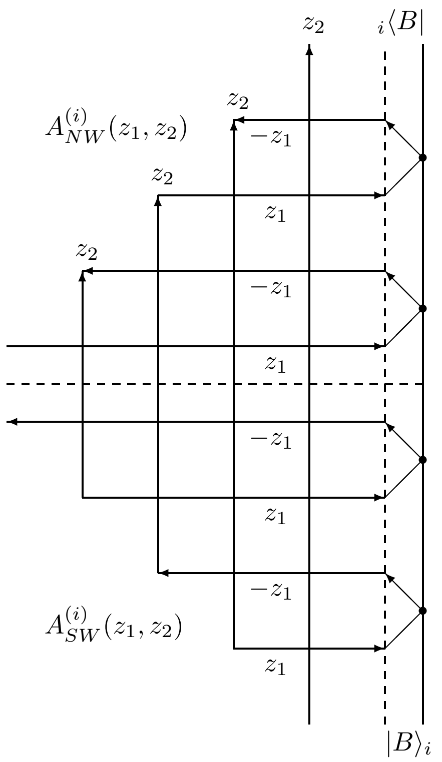
Following [9] we conjecture the partition function $Z_{LM}^{(i)}(z_1, z_2)$ of this model behaves in the thermodynamic limit $L, M \to \infty$ as

$$\begin{aligned} \log Z_{LM}^{(i)}(z_1, z_2) \quad &\sim \quad LM \left( \log \mu^{(i)}(z_1 - z_2) + \log \mu^{(i)}(z_1 + z_2) \right) \\ &+ \quad M \left( \log \nu^{(i)}(z_1) + \log \nu^{(i)}(-z_1 - \tfrac{n}{2}w) \right). \end{aligned} \tag{4.1}$$

Here $\mu^{(i)}(z)$ is the partition function per cite for the bulk theory, and $\nu^{(i)}(z)$ is the that per boundary cite, which are normalized as follows:

$$\mu^{(i)}(z) = 1, \quad \nu^{(0)}(z) = 1, \quad \nu^{(m)}(z) = 1, \text{if } n \text{ is even.} \tag{4.2}$$

Next we consider the boundary CTM lattice as below:




The inhomogeneous CTM lattice split into four sections.

We denote the SW and NW corner transfer matrices by $A_{SW}^{(i)}(z_1, z_2)$ and $A_{NW}^{(i)}(z_1, z_2)$, respectively;

and also denote the upper and lower lines of $K(z)$ by ${}_i\langle B|$ and $|B\rangle_i$, respectively. Let

$$\begin{array}{rcl}\mathcal{H}^{(i)} & := & \{\cdots \otimes v_{p(3)} \otimes v_{p(2)} \otimes v_{p(1)} | p(j) \in \mathbb{Z}_n, p(j) = i+1-j \ (\mathrm{mod}\ n) \text{ for } j \gg 1\}, \\ \bar{\mathcal{H}}^{(i)} & := & \{\cdots \otimes v_{p(3)} \otimes v_{p(2)} \otimes v_{p(1)} | p(j) \in \mathbb{Z}_n, p(j) = i \ (\mathrm{mod}\ n) \text{ for } j \gg 1\},\end{array} \tag{4.3}$$

and $\mathcal{H}^{*(i)}$ and $\bar{\mathcal{H}}^{*(i)}$ be their dual spaces. Then in the infinite lattice limit we conclude that $|B\rangle_i \in \bar{\mathcal{H}}$, ${}_i\langle B| \in \bar{\mathcal{H}}^{*(i)}$, and

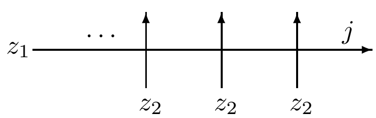
$$\begin{array}{rl}A^{(i)}_{SW}(z_1, z_2): & \bar{\mathcal{H}}^{(i)} \longrightarrow \mathcal{H}^{(i)}, \\ A^{(i)}_{NW}(z_1, z_2): & \mathcal{H}^{(i)} \longrightarrow \bar{\mathcal{H}}^{*(i)}.\end{array} \tag{4.4}$$

The partition function is given as follows:

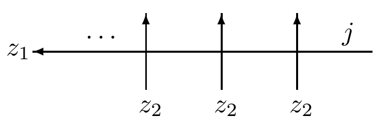
$$Z^{(i)}(z_1, z_2) = {}_i\langle B| A^{(i)}_{NW}(z_1, z_2) A^{(i)}_{SW}(z_1, z_2) |B\rangle_i. \tag{4.5}$$

## 4.2 Vertex operators

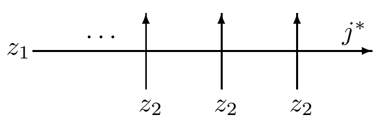
Let us introduce the type I vertex operators



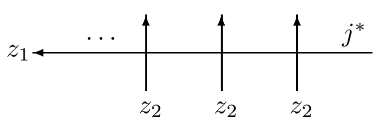

$$= \phi^{j}_{(i-1,i)}(z_1 - z_2) : \mathcal{H}^{(i)} \longrightarrow \mathcal{H}^{(i-1)},$$




$$= \phi_{j}^{(i+1,i)}(z_2 - z_1) : \mathcal{H}^{(i)} \longrightarrow \mathcal{H}^{(i+1)},$$




$$= \phi^{*j}_{(i+1,i)}(z_1 - z_2) : \mathcal{H}^{(i)} \longrightarrow \mathcal{H}^{(i+1)},$$




$$= \phi_{j}^{*(i-1,i)}(z_2 - z_1) : \mathcal{H}^{(i)} \longrightarrow \mathcal{H}^{(i-1)},$$

where the sub/superscripts $(i \pm 1, i)$ specify the spaces intertwined by the vertex operators. We often suppress these sub/superscripts when we have no fear of confusion.

It follows from the Yang–Baxter equation that these vertex operators satisfy the following commutation relations [15, 9]:

$$\begin{array}{rcl}\phi^{j_2}(z_2)\phi^{j_1}(z_1) & = & \displaystyle\sum_{j_1', j_2'} (R^{V_{z_1}, V_{z_2}})^{j_1 j_2}_{j_1' j_2'} \phi^{j_1'}(z_1)\phi^{j_2'}(z_2), \\ \phi^{*j_2}(z_2)\phi^{j_1}(z_1) & = & \displaystyle\sum_{j_1', j_2'} (R^{V_{z_1}, V^*_{z_2}})^{j_1 j_2}_{j_1' j_2'} \phi^{j_1'}(z_1)\phi^{*j_2'}(z_2), \\ \phi^{*j_2}(z_2)\phi^{*j_1}(z_1) & = & \displaystyle\sum_{j_1', j_2'} (R^{V^*_{z_1}, V^*_{z_2}})^{j_1 j_2}_{j_1' j_2'} \phi^{*j_1'}(z_1)\phi^{*j_2'}(z_2).\end{array} \tag{4.6}$$

Furthermore, the unitarity relations for $R$-matrices imply the inversion relation of the vertex operators:

$$\sum_j \phi_j(-z)\phi^j(z) = 1, \qquad \sum_j \phi^*_j(-z)\phi^{*j}(z) = 1. \tag{4.7}$$

From the crossing symmetry we have

$$\phi^{*j}(z) = \phi_j(-z - \tfrac{n}{2}w), \qquad \phi^*_j(-z) = \phi^j(z - \tfrac{n}{2}w). \tag{4.8}$$

Using these vertex operators, the transfer matrix for the semi-infinite lattice is defined as follow:

$$\begin{aligned} T_B(z_1,z_2) &= \sum_{j,k} \phi_j(z_1+z_2)K^j_k(z_1)\phi^k(z_1-z_2) \\ &= \sum_{j,k} \phi^{*j}(-z_1 - \tfrac{n}{2}w - z_2)K^j_k(z_1)\phi^k(z_1-z_2). \end{aligned} \tag{4.9}$$

If the $i$-th vauuam states $|\mathrm{vac}\rangle_i$ and ${}_i\langle\mathrm{vac}|$ satisfy the following reflection properties:

$$\begin{aligned} \sum_k K^j_k(z)\phi^k(z)|\mathrm{vac}\rangle_i &= \nu^{(i)}(z)\phi^j(-z)|\mathrm{vac}\rangle_i, \\ {}_i\langle\mathrm{vac}|\sum_k \phi_k(z)K^k_j(z) &= \nu^{(i)}(z)\,{}_i\langle\mathrm{vac}|\phi_j(-z), \end{aligned} \tag{4.10}$$

these vacuums are the eigenstates of $T_B(z,0)$ associated with the eigenvalues $\nu^{(i)}(z)$, respectively:

$$T_B(z,0)|\mathrm{vac}\rangle_i = \nu^{(i)}(z)|\mathrm{vac}\rangle_i, \qquad {}_i\langle\mathrm{vac}|T_B(z,0) = \nu^{(i)}(z)\,{}_i\langle\mathrm{vac}|.$$

For $n=2$, it is suffices to consider only two types vertex operators $\phi^j(z)$ and $\phi_j(z)$ because of $\phi^{*j}(z) = \phi_{1-j}(-z-w)$ and $\phi^*_j(z) = \phi^{1-j}(-z-w)$ [9]. Furthermore, from $T_B(z_1,z_2) = T_B(-z_1-w,z_2)$ for $n=2$, we have

$$\begin{aligned} & \sum_{j,k} \phi^{1-j}(-z_1-w-z_2)K^j_k(z_1)\phi^k(z_1-z_2) \\ = & \sum_{j',k'} \phi^{1-k'}(z_1-z_2)K^{k'}_{j'}(-z_1-w)\phi^{j'}(-z_1-w-z_2) \\ = & \sum_{\substack{j,k\\ j',k'}} R(-2z_1-w)^{j'\,1-k'}_{1-j\;k}\,\phi^{1-j}(-z_1-w-z_2)\phi^k(z_1-z_2)K^{j'}_{k'}(-z_1-w), \end{aligned} \tag{4.11}$$

which implies the boundary crossing symmetry (3.16).

The crucial point in (4.11) consists in the self-duality $\phi^*_j(z) = \phi^{1-j}(z)$ for $n=2$. Thus the boundary crossing symmetry (3.16) does not have a simple generalization for $n>2$. We should rather regard the RHS of (3.16) for general $n$ as the definition of the dual $K$-matrix. In order to see that, let us repeat the reduction (4.11) for general $n$. Using eqs. (4.8), (4.10), (4.6) and (4.7) we have

$$\begin{aligned} \nu^{(i)}(z) &= \sum_{j',k'} {}_i\langle\mathrm{vac}|\phi^{*k'}(-z-\tfrac{n}{2}w)K^{k'}_{j'}(z)\phi^{j'}(z)|\mathrm{vac}\rangle_i \\ &= \sum_{\substack{j,k\\ j',k'}} {}_i\langle\mathrm{vac}|\phi^j(z)(R^{V_z,V^*_{-z-nw/2}})^{j'k'}_{jk}K^{k'}_{j'}(z)\phi^{*k}(-z-\tfrac{n}{2}w)|\mathrm{vac}\rangle_i \\ &= \sum_{\substack{j,k\\ j',k'}} {}_i\langle\mathrm{vac}|\phi^*_j(-z-\tfrac{n}{2}w)(R^{V_z,V^*_{-z-nw/2}})^{j'k'}_{jk}K^{k'}_{j'}(z)\phi^{*k}(-z-\tfrac{n}{2}w)|\mathrm{vac}\rangle_i. \end{aligned}$$

Thus, if we define the dual $K$-matrix by

$$K^*(-z - \tfrac{n}{2}w)^j_k := \sum_{j',k'} (R^{V_z, V^*_{-z-nw/2}})^{j'k'}_{jk} K(z)^{k'}_{j'}, \tag{4.12}$$

then the following dual reflection properties hold:

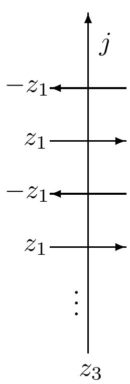
$$\begin{aligned} \sum_k K^*(z)^j_k \phi^{*k}(z)|\mathrm{vac}\rangle_i &= \nu^{(i)}(-z - \tfrac{n}{2}w)\phi^{*j}(-z)|\mathrm{vac}\rangle_i, \\ {}_i\langle\mathrm{vac}| \sum_k \phi^*_k(z) K^*(z)^k_j &= \nu^{(i)}(-z - \tfrac{n}{2}w)\, {}_i\langle\mathrm{vac}|\phi^*_j(-z). \end{aligned} \tag{4.13}$$

The associativity condition of the algebra (4.6) and (4.13) implies the reflection equations involving $K^*$-matrices:

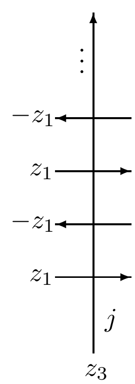
$$\begin{aligned} K_2(z_2) R_{21}^{V_{z_2}, V^*_{-z_1}} K^*_1(z_1) R_{12}^{V^*_{z_1}, V_{z_2}} &= R_{21}^{V_{-z_2}, V^*_{-z_1}} K^*_1(z_1) R_{12}^{V^*_{z_1}, V_{-z_2}} K_2(z_2), \\ K^*_2(z_2) R_{21}^{V^*_{z_2}, V^*_{-z_1}} K^*_1(z_1) R_{12}^{V^*_{z_1}, V^*_{z_2}} &= R_{21}^{V^*_{-z_2}, V^*_{-z_1}} K^*_1(z_1) R_{12}^{V^*_{z_1}, V^*_{-z_2}} K^*_2(z_2). \end{aligned} \tag{4.14}$$

## 4.3 Derivation of the reflection properties

In this subsection we derive the reflection properties (4.10,4.13). For that purpose we introduce following further two types of vertex operators:



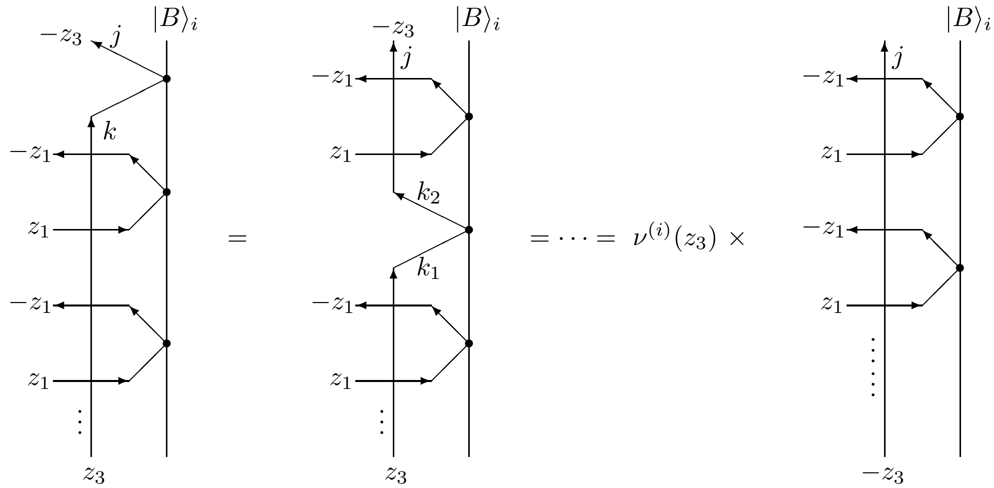

$= \varphi^j_{(i-1,i)}(z_1, z_3) : \bar{\mathcal{H}}^{(i)} \longrightarrow \bar{\mathcal{H}}^{(i-1)}$,




$= \varphi_j^{(i-1,i)}(z_1, z_3) : \bar{\mathcal{H}}^{(*i)} \longrightarrow \bar{\mathcal{H}}^{(*i-1)}$.

where the sub/superscripts $(i \pm 1, i)$ specify the spaces intertwined by the vertex operators. Hereafter we also suppress these sub/superscripts.

From the reflection equation (3.1)



we have the following relation:

$$\sum_k K(z_3)_k^j \varphi^k(z_1,z_3)|B\rangle_i \quad = \quad \nu^{(i)}(z_3)\varphi^j(z_1,-z_3)|B\rangle_i. \tag{4.15}$$

By similar argument we have

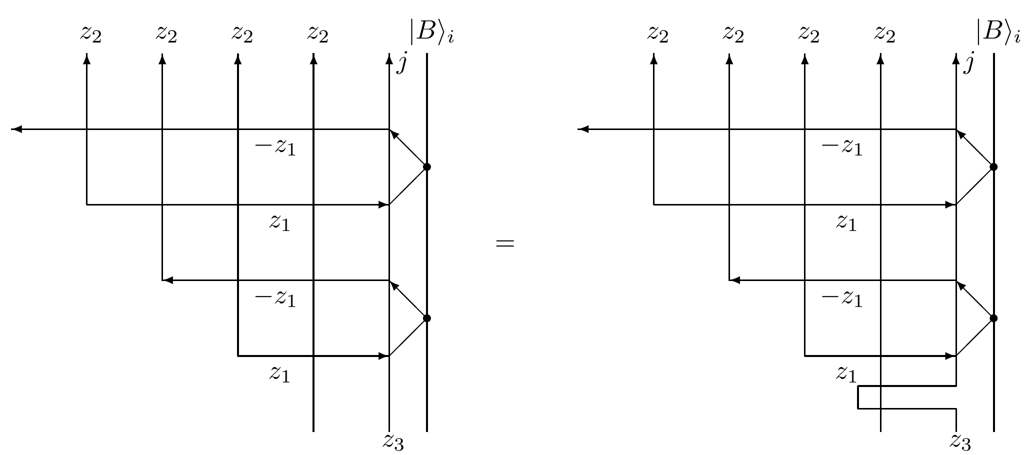
$$\sum_k {}_i\langle B|\varphi_k(z_1,-z_3)K(z_3)_j^k \quad = \quad \nu^{(i)}(z_3)_i\langle B|\varphi_j(z_1,z_3), \tag{4.16}$$

Furthermore, we have the relations

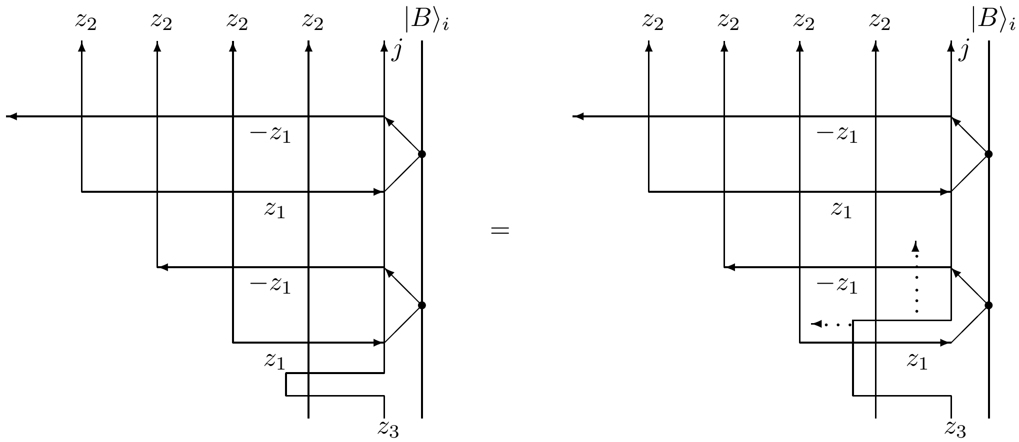
$$A_{SW}^{(i-1)}(z_1,z_2)\varphi^j(z_1,z_3)|B\rangle_i \quad = \quad \phi^j(z_3-z_2)A_{SW}^{(i)}(z_1,z_2)|B\rangle_i, \tag{4.17}$$

$${}_i\langle B|\varphi_j(z_1,z_3)A_{NW}^{(i-1)}(z_1,z_2) \quad = \quad {}_i\langle B|A_{NW}^{(i)}(z_1,z_2)\phi_j(z_2-z_3). \tag{4.18}$$

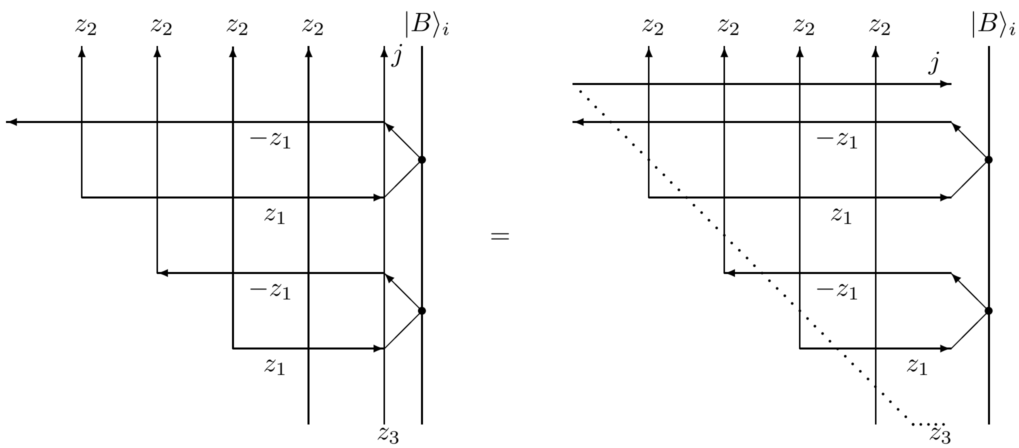
These are based on the unitarity and Yang-Baxter relation of $R$-matrix in the thermodynamic limit. The unitarity (2.28) allows us to obtain



Using the Yang–Baxter equation (2.8) we get




By successive use of the YBE and the unitarity we can bring the line associated with the spectral parameter $z_3$ to the directions pointed by dotted lines in the above figure as far as we like. Thus we find




Those manipulation implies (4.17) because the contribution of Boltzmann weights along the tail graphically represented in the figure by the dotted line is unity in the thermodynamic limit. The relation (4.18) can be similarly obtained.

Applying $A^{(i-1)}_{SW}(z_1, z_2)$ (resp. $A^{(i-1)}_{NW}(z_1, z_2)$) from the left (resp. right) to both sides of (4.15) (resp. (4.16)) and using (4.17) (resp. (4.18)) we obtain

$$\sum_k K(z_3)^j_k \phi^k(z_3 - z_2) A^{(i)}_{SW}(z_1, z_2)|B\rangle_i = \nu^{(i)}(z_3)\phi^j(-z_3 - z_2) A^{(i)}_{SW}(z_1, z_2)|B\rangle_i, \tag{4.19}$$

$$\sum_k {}_i\langle B|A^{(i)}_{NW}(z_1, z_2)\phi_k(z_2 + z_3) K(z_3)^k_j = \nu^{(i)}(z_3)\,{}_i\langle B|A^{(i)}_{NW}(z_1, z_2)\phi_j(z_2 - z_3). \tag{4.20}$$

Taking account of (4.19) and (4.20) with (4.10) we find the following identification

$$|\mathrm{vac}\rangle_i = A^{(i)}_{SW}(z_1, z_2 = 0)|B\rangle_i, \quad {}_i\langle\mathrm{vac}| = {}_i\langle B|A^{(i)}_{SW}(z_1, z_2 = 0). \tag{4.21}$$

From the identification (4.21) and the definition of the dual $K$-matrix (4.12) we obtain

$$\sum_k K^*(z_3)^j_k \phi^{*k}(z_3-z_2) A^{(i)}_{SW}(z_1,z_2)|B\rangle_i = \nu^{(i)}(-z_3-\tfrac{n}{2}w)\phi^{*j}(-z_3-z_2)A^{(i)}_{SW}(z_1,z_2)|B\rangle_i, \quad (4.22)$$

$$\sum_k {}_i\langle B|A^{(i)}_{NW}(z_1,z_2)\phi^*_k(z_2+z_3)K^*(z_3)^k_j = \nu^{(i)}(-z_3-\tfrac{n}{2}w)_i\langle B|A^{(i)}_{NW}(z_1,z_2)\phi^*_j(z_2-z_3). \quad (4.23)$$

# 5 Correlation functions and difference equations

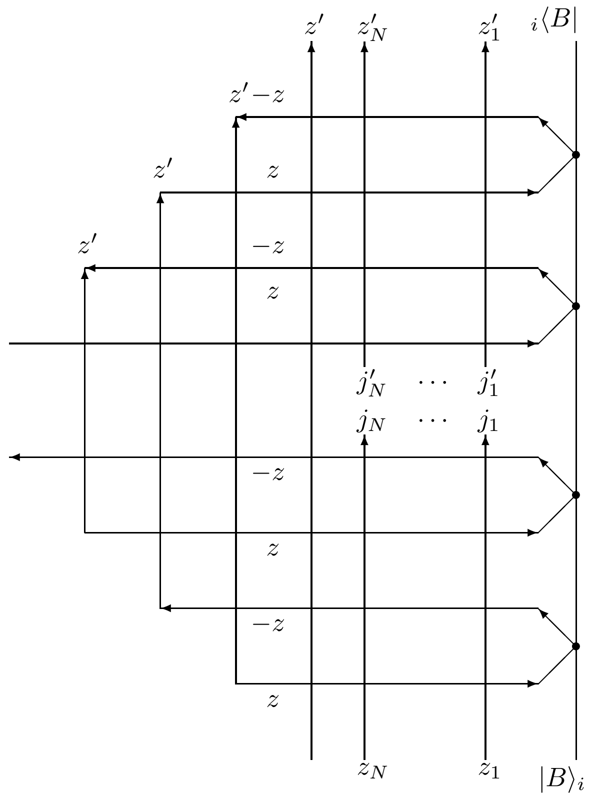
The relations appeared in the previous section are not rigorous because all the objects are defined on the infinite lattice. Nevertheless we assume that eqs. (4.1–4.23) are exactly correct on the basis of the CTM bootstrap method, which is supported by some numerical calculations [14] and consistency with the vertex operator method [15].

## 5.1 Local state probabilities

Let us consider the correlation function on the dislocated CTM lattice

$$G^{(i)}_N(z,z'|z'_1,\cdots,z'_N,z_N,\cdots,z_1)^{j'_1,\cdots j'_N,j_N,\cdots,j_1}$$

$=$




Thanks to (4.17) and (4.18) we have

$$\begin{aligned} & G^{(i)}_N(z,z'|z'_1,\cdots,z'_N,z_N,\cdots,z_1)^{j'_1,\cdots,j'_N,j_N,\cdots,j_1} \\ = & {}_i\langle B|A^{(i)}_{SW}(z,z')\phi_{j'_1}(z'-z'_1)\cdots\phi_{j'_N}(z'-z'_N)\phi^{j_N}(z_N-z')\cdots\phi^{j_1}(z_1-z')A^{(i)}_{SW}(z,z')|B\rangle_i. \end{aligned} \quad (5.1)$$

Thus the correlation function $G_N^{(i)}(z,z'|z'_1,\cdots,z'_N,z_N,\cdots,z_1)^{j'_1,\cdots,j'_N,j_N,\cdots,j_1}$ normalized by the partition function (4.5) is called the $N$-point local state probability of the boundary Belavin model if we set $z_l=z'_l=z'=0$, $j_l=j'_l$ $(1\leqq l\leqq N)$. Owing to the unitarity (4.7) we have

$$Z^{(i)}(z_1,z_2)=\sum_{j_1,\cdots j_N} G_N^{(i)}(z,z'|z_1,\cdots,z_N,z_N,\cdots,z_1)^{j_1,\cdots,j_N,j_N,\cdots,j_1}. \tag{5.2}$$

Thus we obtain the expression of the $n$-point local state probability:

$$P_N^{(i)}(j_1,\cdots,j_N)=\frac{G_N^{(i)}(z,0|0,\cdots,0)^{j_1,\cdots,j_N,j_N,\cdots,j_1}}{\sum\limits_{j_1,\cdots j_N} G_N^{(i)}(z,0|0,\cdots,0)^{j_1,\cdots,j_N,j_N,\cdots,j_1}} \tag{5.3}$$

## 5.2 Boundary analogue of the quantum Knizhnik–Zamolodchikov equation

Only for $n=2$, the $N$-point function (5.1) is reduced to the following $2N$-point function of the form

$$\begin{aligned}&F_{2N}^{(i)}(z,z'|y_1,\cdots,y_N,z_N,\cdots,z_1)^{j'_1,\cdots,j'_N,j_N,\cdots,j_1}\\ =\;& {}_i\langle B|A_{SW}^{(i)}(z,z')\phi^{k_1}(y_1-z')\cdots\phi^{k_N}(y_N-z')\phi^{j_N}(z_N-z')\cdots\phi^{j_1}(z_1-z')A_{SW}^{(i)}(z,z')|B\rangle_i,\end{aligned} \tag{5.4}$$

by putting $y_l=z'_l-w$ and $k_l=1-j'_l$ for $1\leqq l\leqq N$ [9].

It is nothing to do with any local state probabilities for $n>2$, however, we can consider the correlation function of (5.4)-type:

$$\begin{aligned}F_N^{(i)}(z,z'|z_1,\cdots,z_N) &= \sum_{j_1,\cdots,j_N} v_{j_1}\otimes\cdots\otimes v_{j_N}F_N^{(i)}(z,z'|z_1,\cdots,z_N)^{j_1,\cdots,j_N},\\ F_N^{(i)}(z,z'|z_1,\cdots,z_N)^{j_1,\cdots,j_N} &= {}_i\langle B|A_{SW}^{(i)}(z,z')\phi^{j_1}(z_1-z')\cdots\phi^{j_N}(z_N-z')A_{SW}^{(i)}(z,z')|B\rangle_i.\end{aligned} \tag{5.5}$$

Here we assume that $N\equiv 0 \bmod n$ for simplicity.

From the same discussion as in [16, 17], we obtain

**Proposition 5.1** *The correlation function (5.5) satisfies the following relations:*

1.*R-matrix symmetry*:

$$P_{j\,j+1}F_N^{(i)}(z,z'|\cdots,z_{j+1},z_j,\cdots) = R_{j\,j+1}^{V_{z_j},V_{z_{j+1}}}F_N^{(i)}(z,z'|\cdots,z_j,z_{j+1},\cdots), \tag{5.6}$$

2.*Reflection property I* :

$$K_N(z_N)F_N^{(i)}(z,z'|z_1,\cdots,z_{N-1},z_N) = \nu^{(i)}(z_N)F_N^{(i)}(z,z'|z_1,\cdots,z_{N-1},-z_N), \tag{5.7}$$

3.*Reflection property II* :

$$\hat{K}_1(z_1)F_N^{(i)}(z,z'|z_1,z_2,\cdots,z_N) = \nu^{(i)}(z_1)F_N^{(i)}(z,z'|-z_1-nw,z_2,\cdots,z_N), \tag{5.8}$$

*where*

$$\hat{K}(z)v_k=\sum_j v_jK^*(-z-\tfrac{n}{2}w)_j^k.$$

[Proof] The first equation (5.6) follows from the commutation relation (4.6), while the second one (5.7) follows from (4.19). Finally, from the crossing relation (4.8) and (4.23)

$$
\begin{aligned}
& \hat{K}_1(z_1)F_N^{(i)}(z,z'|z_1,z_2,\cdots,z_N) \\
= & \sum_{j'_1,j_1,\cdots,j_N} v_{j_1}\otimes\cdots\otimes v_{j_N}\,{}_i\langle B|A_{SW}^{(i+N)}(z,z')\phi^*_{j'_1}(z'-z_1-\tfrac{n}{2}w)\cdots A_{SW}^{(i)}(z,z')|B\rangle_i K_1^*(-z_1-\tfrac{n}{2}w)_{j_1}^{j'_1} \\
= & \nu^{(i)}(z_1)\sum_{j_1,\cdots,j_N} v_{j_1}\otimes\cdots\otimes v_{j_N}\,{}_i\langle B|A_{SW}^{(i+N)}(z,z')\phi^*_{j_1}(z'+z_1+\tfrac{n}{2}w)\cdots A_{SW}^{(i)}(z,z')|B\rangle_i,
\end{aligned}
$$

we obtain the last equation (5.8). □

Owing to the equations (5.6–5.8) we obtain

**Theorem 5.2** *The correlation function (5.5) satisfies the following difference equation:*

$$
\begin{aligned}
T_jF_N^{(i)}(z,z'|z_1,\cdots,z_N) &= R_{jj-1}^{V_{z_j-nw},V_{z_{j-1}}}\cdots R_{j1}^{V_{z_j-nw},V_{z_1}}\hat{K}_j(-z_j) \\
&\times R_{1j}^{V_{z_1},V_{-z_j}}\cdots R_{j-1j}^{V_{z_{j-1}},V_{-z_j}}R_{j+1j}^{V_{z_{j+1}},V_{-z_j}}\cdots R_{Nj}^{V_{z_N},V_{-z_j}} \\
&\times K_j(z_j)R_{jN}^{V_{z_j},V_{z_N}}\cdots R_{jj+1}^{V_{z_j},V_{z_{j+1}}}F_N^{(i)}(z,z'|z_1,\cdots,z_N),
\end{aligned}
\tag{5.9}
$$

*where*

$$T_jf(z,z'|z_1,\cdots,z_j,\cdots,z_N)=f(z,z'|z_1,\cdots,z_j-nw,\cdots,z_N).$$

Using the crossing symmetries we have another expression of the correlation function on the dislocated CTM lattice for general $n\geqq 2$:

$$
\begin{aligned}
& G_N^{(i)}(z,z'|z_1^*,\cdots,z_N^*,z_N,\cdots,z_1)^{j'_1,\cdots,j'_N,j_N,\cdots,j_1} \\
= & {}_i\langle B|A_{SW}^{(i)}(z,z')\phi^{*j'_1}(z_1^*-z')\cdots\phi^{*j'_N}(z_N^*-z')\phi^{j_N}(z_N-z')\cdots\phi^{j_1}(z_1-z')A_{SW}^{(i)}(z,z')|B\rangle_i,
\end{aligned}
\tag{5.10}
$$

where $z_l^*=z_l'-\frac{n}{2}w$ for $1\leqq l\leqq N$. We thus introduce the $V^{*\otimes n}\otimes V^{\otimes n}$-valued correlation function

$$
\begin{aligned}
& G_N^{(i)}(z,z'|z_1^*,\cdots,z_N^*,z_N,\cdots,z_1) \\
= & \sum_{\substack{j_1,\cdots,j_N\\ j'_1,\cdots j'_n}} v^*_{j'_1}\otimes\cdots\otimes v^*_{j'_N}\otimes v_{j_N}\otimes\cdots\otimes v_{j_1}G_N^{(i)}(z,z'|z_1^*,\cdots,z_N^*,z_N,\cdots,z_1)^{j'_1,\cdots,j'_N,j_N,\cdots,j_1}.
\end{aligned}
\tag{5.11}
$$

Let us describe the $R$-matrix symmetry corresponding to (5.6).

**Proposition 5.3** *Let*

$$
\begin{aligned}
& G_N^{(\sigma i)}(z,z'|x_{\sigma(1)},\cdots,x_{\sigma(2N)}) \\
= & \sum_{\substack{j_1,\cdots,j_N\\ j'_1,\cdots j'_n}} v^*_{j'_1}\otimes\cdots\otimes v^*_{j'_N}\otimes v_{j_N}\otimes\cdots\otimes v_{j_1}G_N^{(i)}(z,z'|x_{\sigma(1)},\cdots,x_{\sigma(2N)})^{k_{\sigma(1)},\cdots,k_{\sigma(N)}}, \\
& G_N^{(\sigma i)}(z,z'|x_{\sigma(1)},\cdots,x_{\sigma(2N)})^{k_{\sigma(1)},\cdots,k_{\sigma(2N)}} \\
= & {}_i\langle B|A_{SW}^{(i)}(z,z')\Phi^{\sigma(1)}\cdots\Phi^{\sigma(2N)}A_{SW}^{(i)}(z,z')|B\rangle_i.
\end{aligned}
\tag{5.12}
$$

*Here $\sigma$ be the permutation of $(1,\cdots,2N)$, and*

$$
x_l=\begin{cases} z_l^*=z_l'-\frac{n}{2}w, & (1\leqq l\leqq N); \\ z_{2N+1-l}, & (N+1\leqq l\leqq 2N); \end{cases}
\qquad
k_l=\begin{cases} j_l', & (1\leqq l\leqq N); \\ j_{2N+1-l}, & (N+1\leqq l\leqq 2N); \end{cases}
$$

*and*

$$\Phi^l = \begin{cases} \phi^{*k_l}(x_l - z'), & (1 \leqq l \leqq N); \\ \phi^{k_l}(x_l - z'), & (N+1 \leqq l \leqq 2N). \end{cases}$$

*Then the following R-matrix symmetry holds:*

$$G_N^{(\sigma_j i)}(z, z'|\cdots, x_{\sigma(j+1)}, x_{\sigma(j)}, \cdots) = R_{\sigma(j),\sigma(j+1)}^{V^{\sigma(j)}, V^{\sigma(j+1)}} G_N^{(\sigma i)}(z, z'|\cdots, x_{\sigma(j)}, x_{\sigma(j+1)}, \cdots), \tag{5.13}$$

*where*

$$V^l = \begin{cases} V_{x_l}^* & (1 \leqq l \leqq N); \\ V_{x_l} & (N+1 \leqq l \leqq 2N); \end{cases}$$

*and $\sigma_j$ is the permutation of $(1, \cdots, 2N)$ obtained from $\sigma$ by transposing $\sigma(j)$ and $\sigma(j+1)$.*

The reflection properties can be similarly shown as before:

**Proposition 5.4** *The following relations holds:*

$$\begin{aligned} K_{2N}(z_1) G_N^{(\pi\, i)}(z, z'|\cdots, z_1) &= \nu^{(i)}(z_1) G_N^{(\pi\, i)}(z, z'|\cdots, -z_1), & (5.14) \\ \hat{K}_{2N}(z_1) G_N^{(\rho\, i)}(z, z'|z_1, \cdots) &= \nu^{(i)}(z_1) T_1 G_N^{(\rho\, i)}(z, z'|-z_1, \cdots), & (5.15) \\ \hat{K}_1^*(z_1^*) G_N^{(\varsigma\, i)}(z, z'|z_1^*, \cdots) &= \nu^{(i)}(-z_1^* - \tfrac{n}{2}w) T_1 G_N^{(\varsigma\, i)}(z, z'|-z_1^*, \cdots), & (5.16) \\ K_1^*(z_1^*) G_N^{(\tau\, i)}(z, z'|\cdots, z_1') &= \nu^{(*i)}(-z_1^* - \tfrac{n}{2}w) G_N^{(\tau\, i)}(z, z'|\cdots, -z_1^*), & (5.17) \end{aligned}$$

*Here,*

$$\hat{K}^*(z) v_k^* = \sum_j v_j^* K(-z - \tfrac{n}{2}w)_j^k,$$

*and $\pi$, $\rho$, $\varsigma$, $\tau \in \mathfrak{S}_{2N}$ such that*

$$\pi(2N) = 2N, \qquad \rho(1) = 2N, \qquad \varsigma(1) = 1, \qquad \tau(2N) = 1.$$

[Proof] The relation (5.13) is evident from the commutation relations (4.6). The last two (5.16) and (5.17) follow from (4.20), (4.8) and (4.22). □

From Propositions 5.3 and 5.4, we have

**Theorem 5.5** *Let $V_1^l = V_{-x_l}$, $V_2^l = V_{x_l - nw}$. Then the following difference equations holds*

$$\begin{aligned} T_l G_N^{(i)}(z, z'|x_1, \cdots, x_{2N}) &= R_{l\, l-1}^{V_2^l, V^{l-1}} \cdots R_{l\, 1}^{V_2^l, V^1} \hat{K}_l^*(-x_l) \\ &\times R_{1\, l}^{V^1, V_1^l} \cdots R_{l-1\, l}^{V^{l-1}, V_1^l} R_{l+1\, l}^{V^{l+1}, V_1^l} \cdots R_{2N\, l}^{V^{2N}, V_1^l} \\ &\times K_l^*(x_l) R_{l\, 2N}^{V_1^l, V^{2N}} \cdots R_{l\, l+1}^{V_1^l, V^{l+1}} G_N^{(i)}(x_1, \cdots, x_{2N}), \end{aligned} \tag{5.18}$$

*for $1 \leqq l \leqq N$, and*

$$\begin{aligned} T_l G_N^{(i)}(z, z'|x_1, \cdots, x_{2N}) &= R_{l\, l-1}^{V_2^l, V^{l-1}} \cdots R_{l\, 1}^{V_2^l, V^1} \hat{K}_l(-x_l) \\ &\times R_{1\, l}^{V^1, V_1^l} \cdots R_{l-1\, l}^{V^{l-1}, V_1^l} R_{l+1\, l}^{V^{l+1}, V_1^l} \cdots R_{2N\, l}^{V^{2N}, V_1^l} \\ &\times K_l(x_l) R_{l\, 2N}^{V_1^l, V^{2N}} \cdots R_{l\, l+1}^{V_1^l, V^{l+1}} G_N^{(i)}(x_1, \cdots, x_{2N}), \end{aligned} \tag{5.19}$$

*for $N+1 \leqq l \leqq 2N$.*

Theorem 5.5 gives an elliptic generalization of the corresponding difference equations for the boundary $U_q(\widehat{sl_n})$-symmetric model [12].

## 5.3 Boundary spontaneous polarization

Applying the similar argument as in (5.9) to the simplest case $N=1$ we obtain the following difference equations:

$$\begin{aligned} T_1 G_1^{(i)}(z,z'|z_1^*,z_2) &= \hat{K}_1^*(-z_1^*) R_{21}^{V_{z_2},V^*_{-z_1^*}} K_1^*(z_1^*) R_{12}^{V^*_{z_1^*},V_{z_2}} G_1^{(i)}(z,z'|z_1^*,z_2), \\ T_2 G_1^{(i)}(z,z'|z_1^*,z_2) &= R_{21}^{V_{z_2-nw},V^*_{z_1^*}} \hat{K}_2(-z_2) R_{12}^{V^*_{z_1^*},V_{-z_2}} K_2(z_2) G_1^{(i)}(z,z'|z_1^*,z_2), \end{aligned} \tag{5.20}$$

where $z_1^* = z_1 - \frac{n}{2}w$. It is difficult to get each element of $G_1^{(i)}(z,z'|z_1,z_2)$, however, it is possible to obtain the expression of the following sums:

$$P_m^{(i)}(z,z'|z_1,z_2) = \sum_{j=0}^{n-1} \omega^{mj} G_1^{(i)}(z,z'|z_1 - \tfrac{n}{2}w, z_2)^{jj}. \tag{5.21}$$

Note that the boundary spontaneous polarization as the vacuum expectation value of the operator $g$ at boundary is expressed in terms of (5.21) as follows:

$$\langle g \rangle^{(i)} = \left. \frac{P_1^{(i)}(z,z'=0|z_1,z_2)}{P_0^{(i)}(z,z'=0|z_1,z_2)} \right|_{z_1=z_2=z'}. \tag{5.22}$$

Now we restrict ourselves to the free boundary condition $r \to 1$ for simplicity. Since $\lim_{r\to 1} \mathcal{K}(0) \neq \mathcal{K}_0$, the initial condition does not hold if we take $\overline{K}(z) = \mathcal{K}_0 \mathcal{K}(z)$. Thus we should regard the $K$-matrix in this limit as $\overline{K}(z) = \mathcal{K}(0)\mathcal{K}(z)$. Under this identification the $K$-matrix behaves as

$$K(z) \longrightarrow k(z) I_n,$$

where $k(z)$ is a scalar function of $z$.

Here we cite the following sum formula from [22][2]

$$\sum_{j=0}^{n-1} \omega^{mj} \frac{\theta^{(j)}(z+w)}{\theta^{(j)}(w)} = n \frac{h((z-m)/n + w) \prod_{l\neq m} h((-z+l)/n)}{h(w) \prod_{l \neq 0} h(l/n)}, \tag{5.23}$$

Then we see the dual $K$-matrix in the free boundary limit $r \to 1$ behaves as

$$K^*(z - \tfrac{n}{2}w) \longrightarrow k(-z) f_0(u^2 q^n) I_n,$$

where

$$\begin{aligned} f_m(u) &:= \sum_{j=0}^{n-1} \omega^{mj} R(z)_{0j}^{j0} \\ &= \frac{1}{\bar{\kappa}(u)} \frac{(\omega^{-m} q^2 u^{-2/n}; t^2)_\infty (t^2 \omega^m q^{-2} u^{2/n}; t^2)_\infty}{(\omega^m u^{2/n}; t^2)_\infty (t^2 \omega^{-m} u^{-2/n}; t^2)_\infty}. \end{aligned} \tag{5.24}$$

The difference equations (5.20) are therefore reduced to

$$\begin{aligned} T_1 G_1^{(i)}(z,z'|z_1^*,z_2)^{jj} &= f_0(u_1^2 q^n) \sum_{k,l} R_{12}(-z_1 - z_2)_{jk}^{kj} R_{21}(z_2 - z_1)_{kl}^{lk} G_1^{(i)}(z,z'|z_1^*,z_2)^{ll}, \\ T_2 G_1^{(i)}(z,z'|z_1^*,z_2)^{jj} &= f_0(u_2^2 q^n) \sum_{k,l} R_{12}(z_1 - z_2)_{jk}^{kj} R_{21}(-z_1 - z_2)_{kl}^{lk} G_1^{(i)}(z,z'|z_1^*,z_2)^{ll}, \end{aligned} \tag{5.25}$$

[2] Note that there are typographical errors in the formula [22].

where $z_1^* = z_1 - \frac{n}{2}w$, and we use (2.28) and (3.11). Substituting (5.25) into (5.21) we obtain

$$P_m^{(i)}(z, z'|z_1, z_2) = C_m^{(i)} A(u_1)A(u_2)B_m(u_+)B_{-m}(u_-). \tag{5.26}$$

Here $C_m^{(i)}$ is a constant, and $A(u)$ and $B_m(u)$ are solutions to the following difference equations:

$$\frac{A(uq^n)}{A(u)} = f_0(u^2q^n), \qquad \frac{B_m(uq^{-n})}{B_m(u)} = f_m(u). \tag{5.27}$$

By solving these difference equations we obtain

$$A(u) = \psi(u^2)\frac{(q^2u^{4/n}; t^2, q^4)_\infty (q^4u^{-4/n}; t^2, q^4)_\infty}{(t^2u^{4/n}; t^2, q^4)_\infty (t^2q^2u^{-4/n}; t^2, q^4)_\infty}, \tag{5.28}$$

where

$$\psi(u) := g_0(uq^{-n/2})g(u^{-1}q^{n/2}), \qquad g_0(u) := \frac{(q^{2+3n}u^{-2}; t^2, q^{2n}, q^{4n})_\infty (t^2q^{-2+3n}u^{-2}; t^2, q^{2n}, q^{4n})_\infty}{(q^{3n}u^2; t^2, q^{2n}, q^{4n})_\infty (t^2q^{3n}u^2; t^2, q^{2n}, q^{4n})_\infty};$$

and

$$B_m(u) = \varphi(u)\frac{(t^2\omega^m u^{2/n}; t^2)_\infty (t^2\omega^{-m}u^{-2/n}; t^2)_\infty}{(q^2\omega^m u^{2/n}; q^2)_\infty (q^2\omega^{-m}u^{-2/n}; q^2)_\infty}, \tag{5.29}$$

where

$$\varphi(u) := g(uq^{n/2})g(u^{-1}q^{n/2}), \qquad g(u) := \frac{(q^{3n}u^{-2}; t^2, q^{2n}, q^{2n})_\infty (t^2q^{3n}u^{-2}; t^2, q^{2n}, q^{2n})_\infty}{(q^{2+n}u^2; t^2, q^{2n}, q^{2n})_\infty (t^2q^{-2+n}u^2; t^2, q^{2n}, q^{2n})_\infty}.$$

Note that $B_m(u)$ is essentially the same as $G^{(m)}(u)$ in [22], which corresponds to the quantity (5.21) in the bulk theory.

From (5.26) we have

$$\frac{P_1^{(i)}(z, z'=0|z_1, z_2)}{P_0^{(i)}(z, z'=0|z_1, z_2)} = \frac{C_1^{(i)}}{C_0^{(i)}}\frac{B_1(u_+)B_{-1}(u_-)}{B_0(u_+)B_0(u_-)}. \tag{5.30}$$

Taking the low temperature limit $t, q \to 0$, we find that the ratio $C^{(i)}/C_0^{(i)}$ should be equal to $\omega^i$. We therefore obtain the boundary spontaneous polarization from (5.30) and (5.29) by putting $u_+ = u_- = 1$

$$\langle g\rangle^{(i)} = \omega^i \frac{(q^2; q^2)_\infty^4}{(t^2; t^2)_\infty^4}\frac{(t^2\omega; t^2)_\infty^2 (t^2\omega^{-1}; t^4)_\infty^2}{(q^2\omega; q^2)_\infty^2 (q^2\omega^{-1}; q^4)_\infty^2}. \tag{5.31}$$

When $n = 2$ this expression coincides with the previous result obtained in [9]. We also emphasize that the boundary spontaneous polarization for the boundary Belavin model is exactly the square of that for the bulk Belavin model obtained in [22], up to a phase factor.

# 6 Summary and discussion

In this paper we have obtained two non-diagonal solutions of the reflection equation associated with Belavin's $\mathbb{Z}_n$-symmetric elliptic model. Unfortunately, our elliptic $K$-matrix is not connected with the diagonal boundary Boltzmann weights for the $A_{n-1}^{(1)}$-face model [23] but the non-diagonal ones. It is

thus an open problem to obtain the $K$-matrix corresponding to the boundary Boltzmann weights given in [23].

On the basis of the boundary CTM bootstrap we have derived a set of difference equations for correlation functions of the boundary Belavin model. By solving the simplest difference equations, we have obtained the boundary spontaneous polarization of the boundary Belavin model. Our result is consistent with the one given in [9] when $n = 2$. The boundary spontaneous polarization is equal to the square of the bulk spontaneous polarization [22] up to a phase factor. The same phenomena were observed in [8, 9].

In this paper we have shown that correlation functions of the boundary model satisfy the $R$-matrix symmetry and the reflection properties, which are the boundary analogue of Smirnov's first two axioms [18]. It may be interesting to construct integral formulae for correlation functions such that the integrand possesses the determinant structure as in Smirnov's integral [18].

In [13] integral formulae for correlation functions of the boundary $XYZ$ model by using bosonization of vertex operators [35]. In order to obtain the higher $n$ generalization of [13], the construction of free field realization of the boundary Belavin model is required. It is a very hard but important work.

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